

Exercise 1.1

Let the decision variables be $x_1, \dots, x_7$:

(i) $\sum_{i=1}^7 x_i \leq 6$

(ii) $\sum_{i=1}^7 x_i \geq 1$

(iii) $x_1 + x_3 \leq 1$

(iv) $x_4 \leq x_2$

(v) $x_1 = x_5$

(vi) Here it is important to note that the “or” is interpreted as a logical or and **not** an “exclusive or”. $2x_1 + 2x_2 + 2x_3 + x_2 + x_4 + x_5 + x_6 \geq 2$.

An equivalent formulation, we define two extra binary variables y_1 and y_2 , then:

$$x_1 + x_2 + x_3 \geq 1y_1$$

$$x_2 + x_4 + x_5 + x_6 \geq 2y_2$$

$$y_1 + y_2 = 1$$

In the unlikely event that we are looking for an “XOR” formulation, that is, either the first constraint must be fulfilled and the second **not** or vice versa we get a slightly more complicated formulation:

$$x_1 + x_2 + x_3 \geq 1y_1 - My_2$$

$$x_2 + x_4 + x_5 + x_6 \leq 1y_1 + My_2$$

$$x_1 + x_2 + x_3 \leq 0y_2 + My_1$$

$$x_2 + x_4 + x_5 + x_6 \geq 2y_2 - My_1$$

$$y_1 + y_2 = 1$$

Exercise 1.2

Here we assume that x_1 and x_2 are *variables*. The problem gets slightly easier if we assume that they are parameters.

(i)
$$\begin{array}{ll} u = \max & z \\ \text{st.} & z \leq x_1 \\ & z \leq x_2 \\ & z \geq 0 \end{array}$$

$$\begin{aligned}
 u = \min \quad & z \\
 \text{(ii) st.} \quad & x_1 - x_2 \leq z \\
 & x_2 - x_1 \leq z \\
 & z \geq 0
 \end{aligned}$$

And for both (i) and (ii) we have:

$$\begin{aligned}
 x_j &\leq C & \forall j \\
 x_j &\geq 0 & \forall j
 \end{aligned}$$

Exercise 1.5

In order to simplify the description of some of the constraints we define a matrix a_{ikt} based on the t_{ik} . So let $a_{ikt} = 1$ if $t_{ik} = t$.

(i) John has to take exactly 4 courses:

$$\sum_{ik} x_{ik} = 4$$

He only has to take one section of each course:

$$\sum_i x_{ik} \leq 1 \quad \forall k$$

He cannot have more than one course at each time slot:

$$\sum_{i,k} a_{ikt} x_{ik} \leq 1 \quad \forall t$$

Finally we will maximize John's preferences:

$$\max \sum_{i,k} p_{ik} \cdot x_{ik}$$

(ii) No more than two consecutive hours at a time. .

$$\sum_{k,i} (a_{ikt} x_{ik} + a_{ik(t+1)} x_{ik} + a_{ik(t+2)} x_{ik}) \leq 2 \quad \forall t \in \{10, \dots, 19\}$$

(iii) In order for John to start as late as possible define a variable T denoting the starting time of the first class. Then:

$$T \leq t_{ik} x_{ik} + (1 - x_{ik}) 19 \quad \forall i, k,$$

and

$$\max T$$

A Container Problem

(a) In order to solve the container problem we note that the two decisions we have to make are do we want to use a container or not, and if not where do we pour the contents. To reflect this we need two binary (decision) variables:

$$y_i = \begin{cases} 1 & \text{if we use container } i \\ 0 & \text{otherwise} \end{cases}$$

for each container i and

$$x_{ij} = \begin{cases} 1 & \text{if the contents of } i \text{ is poured into } j \\ 0 & \text{otherwise} \end{cases}$$

for any pair i, j of containers.

As we want to use as few containers as possible the objective function we be

$$\min \sum_{i=1}^n y_i$$

First we need to ensure that the total volume of a container is never exceeded

$$\sum_{i=1, i \neq j}^n a_i x_{ij} + a_j y_j \leq v_j y_j \quad \text{for } j = 1, 2, \dots, n$$

And we need to ensure that we do not pour contents from a container we use in the end **and** that the liquid is not split.

$$\sum_{j=1, j \neq i}^n x_{ij} + y_i = 1 \quad \text{for } i = 1, 2, \dots, n$$

As we do not really use x_{ii} for any i we might use x_{ii} as y_i and discard the y -variables. This is not so “beautiful” but brings us down to only one set of variables (in total n^2).