

Exercise 7.1

(I) Since the problem considered is a minimization problem, an upper bound is a feasible solution and hence any upper bound on any node can be used. The best(lowest) such upper bound is 31 (from node 7).

The lower bound is found by propagating lower bounds up in the tree. Node 8 is infeasible and can thus be pruned. The lower bound of node 4 is then 31, since this is the only child of node 4. The lower bound of node 1 is 28 since this is the worst lower bound of its children 3 and 4. The lower bound of node 2 is 27 since this is the worst lower bound of its children. Finally the lower bound of node 0 is 27 since this is the worst lower bound of its children. So the best(highest) lower bound is 27.

(II) Node 8 can be pruned since it is infeasible and node 7 can be pruned since it is optimal. All children of node 4 has then been considered, so it can be pruned. Node 6 can be pruned since its lower bound is worse than the global upper bound(31) - it can never lead to a solution better than 35. Nodes 3 and 5 (and indirectly nodes 0, 1 and 2) has to be explored further.

Exercise 7.3

(I) The items are sorted in non-increasing order of “value per unit” and the items with highest values per units are selected first. One element (denoted r) is selected partially - this is allowed since the LP relaxation is considered.

Is this solution optimal? If it is not, then, since it is an LP problem, a pair of variables must exists, such that reducing one variable with a small number and increasing the other variable would improve the solution. All variables have to be non-negative, so the variable to reduce can only be one of the nonzero variables (i.e. $x_i, 1 \leq i \leq r$). Also the variable to increase can only be one of the variables <1 , (i.e. $x_j, r \leq j \leq n$). But in that case $\frac{c_i}{a_i} \geq \frac{c_j}{a_j}$. Also decreasing x_i by $\Delta > 0$ will allow us to increase x_j by at most $\Delta \frac{a_i}{a_j}$ for a change in the objective of $(-c_i + c_j \frac{a_i}{a_j})\Delta \leq 0$. So it does not pay of to make the change and thus the suggested solution is optimal.

(II) We check that the x_j 's are actually sorted in non-increasing order: $3\frac{2}{5} \geq 3\frac{1}{3} \geq 3\frac{1}{8} \geq 2\frac{3}{7}$.

The branch-and-bound tree is shown in figure 1.

Given (I), the LP solution is then $x_1 = x_2 = 1, x_3 = (12 - (5 + 3))/8 = \frac{1}{2}, x_4 = 0$ and $Z = 39\frac{1}{2}$ corresponding to node 0. We branch on x_3 since it is the only non-integer variable. This gives the two nodes 1 and 2.

Node 1 has the solution $x_3 = 0, x_1 = x_2 = 1, x_4 = \frac{4}{7}$ and $Z = 36\frac{5}{7}$. This gives rise to two new nodes, 3 and 4 with $x_4 = 0$ and $x_4 = 1$ respectively.

Node 2 has the solution $x_3 = 1, x_1 = \frac{4}{5}, x_2 = x_4 = 0$ and $Z = 38\frac{3}{5}$. This gives rise to two new nodes, 5 and 6 with $x_1 = 0$ and $x_1 = 1$ respectively.

Node 3 has the solution $x_3 = x_4 = 0, x_1 = x_2 = 1$ and $Z = 27$. Since the variables are integral, this is a feasible solution and thus a globally valid lower bound. Also the node is pruned since it is optimal.

Node 4 has the solution $x_3 = 0, x_4 = x_1 = 1, x_2 = 0$ and $Z = 34$. Feasible solution - best lower

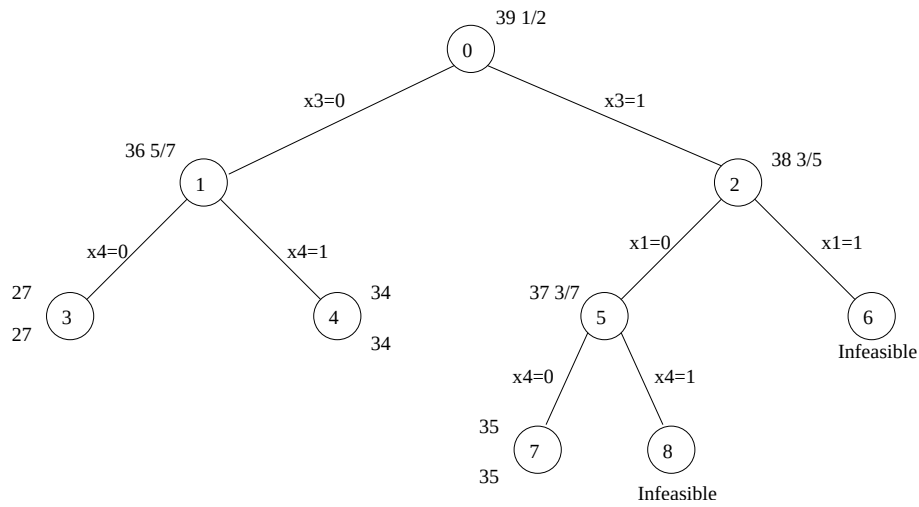


Figure 1: *Branch-and-bound tree for exercise 7.3.*

bound, pruned since it is optimal.

Node 5 has the solution $x_3 = 1, x_1 = 0, x_2 = 1, x_4 = \frac{1}{7}$ and $Z = 37\frac{3}{7}$. This gives rise to two new nodes, 7 and 8 with $x_4 = 0$ and $x_4 = 1$ respectively.

Node 6 has the solution $x_3 = x_1 = 1$, which is infeasible, so it is pruned.

Node 7 has the solution $x_3 = 1, x_1 = x_4 = 0, x_2 = 1$ and $Z = 35$. Feasible solution - best lower bound, pruned since it is optimal.

Node 8 has the solution $x_3 = x_4 = 1$ and $x_1 = 0$, which is infeasible, so it is pruned.

All nodes have been considered and so, since node 7 has the best solution it is optimal. I.e. $x_2 = x_3 = 1, x_1 = x_4 = 0$ and $Z = 35$.

Exercise 7.9

$$\max \quad 5x_1 \quad -7x_2 \quad -10x_3 \quad +3x_4 \quad -5x_5 \quad (1)$$

$$x_1 \quad +3x_2 \quad -5x_3 \quad +x_4 \quad +4x_5 \quad \leq 0 \quad (2)$$

$$-2x_1 \quad -6x_2 \quad +3x_3 \quad -2x_4 \quad -2x_5 \quad \leq -4 \quad (3)$$

$$2x_2 \quad -2x_3 \quad -x_4 \quad +x_5 \quad \leq -2 \quad (4)$$

An infeasibility can be derived for this BIP. It can be derived in a number of different ways. One possible road-map is:

1. Based on the first constraint we see that if x_3 is not 1 all other variables must also be 0 as x_3 is the only one with a negative coefficient in the first constraint, so

$$x_1 \leq x_3, x_2 \leq x_3, x_4 \leq x_3, x_5 \leq x_3$$

Furthermore both x_2 and x_5 can be 1 at the same time, ie.

$$x_2 + x_5 \leq 1$$

2. From the second constraint we see that if x_3 is 1 we need x_2 also to be 1 if we want feasibility, so

$$x_3 \leq x_2$$

3. From the third constraint we can deduct: x_2 must be 0 and x_3 must be 1 otherwise the left hand side cannot be smaller than or equal to 2.
4. Now in total we get: $x_3 \leq x_2$ and $x_2 \leq x_3$ so $x_2 = x_3$. But this contradicts the findings from the third constraint, hence the problem is infeasible.

Alternatively we could use the rules on page 105-106 to simplify the inequalities. Using rule (i) with $x_0 = x_3$, inequality 4 gives the following:

$$x_3 \geq (-2 - 1 \cdot 0 - 2 \cdot 0 - 1 \cdot (-1)) / -2 = \frac{1}{2} \quad (5)$$

So $x_3 = 1$. Likewise using rule (i) with $x_0 = x_2$ and $x_3 = 1$, inequality 4 gives the following:

$$x_2 \leq ((-2 + 2) - 1 \cdot 0 - (-1)) / 2 = \frac{1}{2} \quad (6)$$

So $x_2 = 0$. Inserting x_2 and x_3 in inequality 3 gives the following equation.

$$-2x_1 - 6 \cdot 0 + 3 \cdot 1 - 2x_4 - 2x_5 \leq -4 \Leftrightarrow -2x_1 - 2x_4 - 2x_5 \leq -7 \quad (7)$$

Using rule (iii) on inequality 7, it can be seen that no solution exists:

$$-2 \cdot 1 - 2 \cdot 1 - 2 \cdot 1 > -7 \Leftrightarrow -6 > -7 \Leftrightarrow 6 < 7 \quad (8)$$