

Exercise 5.1

Solved as example 5.1 page 70.

Results: $c = (5, 4, 3, 3)$ and $\sum_{t=1}^4 h_t d_{1t} = 77$.

The optimal solution is (variables not listed are zero): $y_1 = 1, x_1 = 8, y_2 = 1, x_2 = 22, s_2 = 17$ and $s_3 = 4$ with a value of $158 - 77 = 81$.

Exercise 5.3

Starting with the leaves, $H(4) = 4, H(9) = 2, H(10) = 3, H(6) = 0, H(7) = 0, H(11) = 0$ and $H(12) = 3$. I.e. the optimal subtree of nodes with positive label is the node it self, and the optimal subtree of nodes with negative label is the empty subtree at a cost of 0. Now nodes $H(5)$ and $H(8)$ can be calculated, since the optimal solution for all successors are known. $H(5) = \max[0, -6 + 2 + 3] = 0$ and $H(8) = \max[0, 1 + 0 + 3] = 4$. This is continued: $H(2) = \max[0, -1 + 4 + 0 + 0] = 3, H(3) = \max[0, -1 + 0 + 4] = 3$ and the optimal value: $H(1) = \max[0, -2 + 3 + 3] = 4$.

Calculating backwards, the optimal subtree consist of the nodes: 1, 2, 3, 4, 8, 12.

Exercise 5.5

(See page 67-68 for a description of the problem without passage times.)

(I) Let $f_k(j)$ be the shortest time required to go from 1 to j using $\leq k$ arcs (∞ if j cannot be reached). For $k = 1, f_1(j) = \max\{c_{1j}, r_j\}$ if $1j \in A$ otherwise ∞ . The recurrence for $2 \leq k \leq |A|$ is then:

$$f_k(j) = \max\{r_j, \min\{f_{k-1}(j), \min_{i \in V^-(j)} \{c_{ij} + f_{k-1}(i)\}\}\}$$

(II) We need two kinds of variables: t_i which is the time node i is traversed (if i is never traversed it is unspecified) and $x_{ij} = 1$ if arc ij is traversed, 0 otherwise.

The MIP program is given below.

$$\min t_n \tag{1}$$

$$\sum_i x_{1i} = 1 \tag{2}$$

$$\sum_i x_{in} = 1 \tag{3}$$

$$\forall i, i \neq 1, i \neq n : \quad \sum_{j, i \neq j} x_{ij} = \sum_{j, i \neq j} x_{ji} \tag{4}$$

$$\forall j : \quad r_j \leq t_j \tag{5}$$

$$\forall i, j, i \neq j : \quad t_i + c_{ij}x_{ij} - M(1 - x_{ij}) \leq t_j \tag{6}$$

$$\forall i, j, i \neq j : \quad x_{ij} \in \{0, 1\} \tag{7}$$

$$\forall j : \quad t_i \geq 0 \tag{8}$$

Constraint 2 ensures that one arc out of node 1 is selected and constraint 3 ensures that one arc in to node n is selected. Constraints 4 ensure that for all other nodes, either no arc leaves or enters the node or exactly one arc enters and one arc leaves the node. Constraints 5 put a limit on how early node j can be traversed.

Constraints 6 are designed to put a limit on the time node j is visited, if it is on the selected path. If $x_{ij} = 1$, the limit is in effect, i.e. node j cannot be visited earlier than node i is visited plus the time it takes to traverse edge ij . If $x_{ij} = 0$, the term $-M(1 - x_{ij})$ ensures that there is no limit on t_j .

The number of binary variables is $O(m)$ where m is the number of arcs. Likewise the number of constraints is $O(m)$ (assuming $m \geq n$). This is to some extent fine - neither is exponential at least. So one way to go would be to solve the problem by solving the LP-relaxation and hope the optimal solution is integer or at least that the lower bound obtained is good. It is probably not - formulations involving “ M ” are usually not nice and in fact the quality of the lower bound depends on what values of M are used.

Exercise 5.10

$f(S, j)$ denotes the length of a shortest Hamiltonian path starting at node 1 terminating at node j and passing through the remaining nodes of S .

In order to find the shortest Hamiltonian path from 1 to j “via” S we can find the smallest sum of:

- the shortest Hamiltonian path from 1 to a node i via $S \setminus \{j\}$ plus
- the length of the arc from i to j .

This can be formulated as the following recursive formula:

$$f(S, j) = \min_{i \in S, i \neq 1, i \neq j} \{f(S \setminus \{j\}, i) + c_{ij}\}$$

Note that there will be exponentially many recursive calls to determine the necessary values for each stage. Now we just need to define the starting of the recursion.

$$f(\{1, j\}, j) = c_{1j} \quad \forall j$$

And in order to solve the symmetric TSP we need to find $f(V, 1)$ where V is the set of vertices of the problem.