

Solutions to Exercises in LP and Duality

Exercise 8

The problem is solved by Simplex algorithm using the “least row index” pivot row selection rule, also known as the *Dantzig’s Rule*: at each iteration of the algorithm, a nonbasic variable with the largest value in the objective function (in case of a maximization problem) is chosen as the *entering basic variable*. A basic variable, which sets the tightest bound on the entering basic variable is chosen as the *leaving basic variable*.

Starting with nonbasic variables x_1, x_2 and x_3 , $Z = 0$:

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	-4	-2	-1	0	0	0	0	
x_4	1	0	0	1	0	0	1	$x_1 \leq \frac{1}{1} = 1 \quad \leftarrow \min$
x_5	4	1	0	0	1	0	4	$x_1 \leq \frac{4}{4} = 1$
x_6	8	4	1	0	0	1	16	$x_1 \leq \frac{16}{8} = 2$

The nonbasic variable x_1 enters the basic, the basic variable x_4 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	0	-2	-1	4	0	0	4	
x_1	1	0	0	1	0	0	1	none
x_5	0	1	0	-4	1	0	0	$x_2 \leq \frac{0}{1} = 0 \quad \leftarrow \min$
x_6	0	4	1	-8	0	1	8	$x_2 \leq \frac{8}{4} = 2$

The nonbasic variable x_2 enters the basic, the basic variable x_5 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	0	0	-1	-4	2	0	4	
x_1	1	0	0	1	0	0	1	$x_4 \leq \frac{1}{1} = 1 \quad \leftarrow \min$
x_2	0	1	0	-4	1	0	0	none, $x_4 \geq 0$
x_6	0	0	1	8	-4	1	8	$x_4 \leq \frac{8}{-4} = -2$

The nonbasic variable x_4 enters the basic, the basic variable x_1 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	4	0	-1	0	2	0	8	
x_4	1	0	0	1	0	0	1	none
x_2	4	1	0	0	1	0	4	none
x_6	-8	0	$\boxed{1}$	0	-4	1	0	$x_3 \leq \frac{0}{1} = 0 \leftarrow \min$

The nonbasic variable x_3 enters the basic, the basic variable x_6 leaves.

... and so on... until all nonbasic variables have nonnegative coefficients in the objective function.

The *last* Simplex tableaux:

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	4	2	0	0	0	1	16
x_4	1	0	0	1	0	0	1
x_5	4	1	0	0	1	0	4
x_3	8	4	1	0	0	1	16

The solution: $Z = 16$, $x_1 = 0$, $x_2 = 0$, $x_3 = 16$.

There are several rules for choosing the entering variable in the Simplex method. As mentioned, the above iterations are solved with the Dantzig's Rule. Another rule is the *Maximum Improvement Rule*. At each iteration of the algorithm, the minimum ratio test is performed for *each* nonbasic variable. The variable, which gives the best improvement in the objective function from its upper bound, is chosen as the entering basic variable.

BV	x_1	x_2	x_3	x_4	x_5	x_6	RS	UB			Improve Z
Z	-4	-2	-1	0	0	0	0	x_1	x_2	x_3	
x_4	1	0	0	1	0	0	1	$\frac{1}{1} = \boxed{1}$	none	none	$4 \cdot 1 = 4$
x_5	4	1	0	0	1	0	4	$\frac{4}{4} = 1$	$\frac{4}{1} = \boxed{4}$	none	$2 \cdot 4 = 8$
x_6	8	4	$\boxed{1}$	0	0	1	16	$\frac{16}{8} = 2$	$\frac{16}{4} = 4$	$\frac{16}{1} = \boxed{16}$	$1 \cdot 16 = 16 \leftarrow \max$

The nonbasic variable x_3 enters the basic, the basic variable x_6 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	4	2	0	0	0	1	16
x_4	1	0	0	1	0	0	1
x_5	4	1	0	0	1	0	4
x_3	8	4	1	0	0	1	16

The solution: $Z = 16$, $x_1 = 0$, $x_2 = 0$, $x_3 = 16$.

Exercise 9

It is straight-forward to show that problem b) is the dual of problem a), where the coefficients of the objective function of a) correspond to the right side of b) and vice versa, and every column of a) corresponds to the row of b) and vice versa. For further information please refer to e.g. *Introduction to Operations Research* by Hillier & Lieberman.

Solving a) by the Simplex algorithm gives following results:

Starting with nonbasic variables x_1, x_2 and x_3 , $Z = 0$:

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	-1	-1	-1	0	0	0	0	
x_4	0	1	2	1	0	0	1	none
x_5	1	2	1	0	1	0	1	$x_1 \leq \frac{1}{1} = 1$
x_6	2	0	0	0	0	1	1	$x_1 \leq \frac{1}{2} \quad \leftarrow \min$

The nonbasic variable x_1 enters the basic, the basic variable x_6 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	0	-1	-1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	
x_4	0	1	2	1	0	0	1	$x_2 \leq \frac{1}{1} = 1$
x_5	0	2	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$x_2 \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \quad \leftarrow \min$
x_1	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	none

The nonbasic variable x_2 enters the basic, the basic variable x_5 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side	UB
Z	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	
x_4	0	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	$x_2 \leq \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2} \quad \leftarrow \min$
x_2	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$x_2 \leq \frac{1}{4} \cdot 2 = \frac{1}{2}$
x_1	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	none

The nonbasic variable x_3 enters the basic, the basic variable x_4 leaves.

Basic Variable	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1
x_3	0	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$
x_2	0	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	0
x_1	1	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

The optimal solution $Z = 1$ with $x_1 = \frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}$.

Notice that the problem has several optimal solutions with $Z = 1$. For instance, another optimal solution is $x_1 = \frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}$, which is achieved by the Simplex algorithm if x_2 is chosen as the leaving basic variable instead of x_4 in the third iteration.

The optimal dual solution is $Z_D = 1$. It is shown how dual variables are obtained from the final tableaux of the Simplex algorithm in Exercise 10.

Exercise 10

The dual problem to the problem a), which is modified by adding slack variables to obtain the equality in constraints is shown below.

$$\begin{aligned} \min \quad & y_1 + y_2 + y_3 \\ \text{s.t.} \quad & \\ & y_2 + 2y_3 \geq 1 \\ & y_1 + 2y_2 \geq 1 \\ & 2y_1 + y_2 \geq 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

By solving the problem a) with 6 variables (3 original and 3 slack) in CPLEX, the following information is obtained:

Reduced costs: $\bar{c}_1 = 0, \bar{c}_2 = 0, \bar{c}_3 = 0, \bar{c}_4 = -\frac{1}{3}, \bar{c}_5 = -\frac{1}{3}, \bar{c}_6 = -\frac{1}{3}$.

Dual values: $y_1 = \frac{1}{3}, y_2 = \frac{1}{3}, y_3 = \frac{1}{3}$.

There is a direct relationship between the reduced costs and the dual values. Written in vectors: $\bar{\mathbf{c}} = \mathbf{c} - \mathbf{y}\mathbf{A}$. To check the relationship on the above example:

$$\bar{\mathbf{c}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Due to the relationship between the reduced costs and the dual variables, the dual variables can be read directly from the final Simplex tableaux as *coefficients of the nonbasic variables* in the optimal solution (e.g. x_4, x_5, x_6

in this example) *with the opposite sign* if all these variables have zero-one columns in the original coefficient matrix $\mathbf{A}$. In this case, dual variables for the problem a) are $y_1 = -\bar{c}_1 = \frac{1}{3}, y_2 = -\bar{c}_2 = \frac{1}{3}, y_3 = -\bar{c}_3 = \frac{1}{3}$, since the columns corresponding to x_4, x_5, x_6 in the original coefficient matrix are

$$A_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Exercise 11

When the Dat Oil problem from Exercise 7 is solved in CPLEX, the following information is obtained (all values are rounded down): variables x_{ij} and y_i , slacks s_k , reduced costs $\bar{c}_{ij}$ corresponding to x_{ij} and $\bar{c}_i$ corresponding to y_i , and dual variables π_k .

$$\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \\ x_{31} \\ x_{32} \\ x_{33} \\ x_{41} \\ x_{42} \\ x_{43} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3457 \\ 0 \\ 0 \\ 5050 \\ 2454 \\ 0 \\ 4646 \\ 2454 \\ 0 \\ 1846 \\ 543 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \\ s_8 \\ s_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5093 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \bar{c}_{11} \\ \bar{c}_{12} \\ \bar{c}_{13} \\ \bar{c}_{21} \\ \bar{c}_{22} \\ \bar{c}_{23} \\ \bar{c}_{31} \\ \bar{c}_{32} \\ \bar{c}_{33} \\ \bar{c}_{41} \\ \bar{c}_{42} \\ \bar{c}_{43} \\ \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2,02 \\ 0 \\ 0 \\ -0,85 \\ 0 \\ 0 \\ -0,52 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2,20 \\ -1,71 \\ -2,69 \end{bmatrix} \quad \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \\ \pi_8 \\ \pi_9 \end{bmatrix} = \begin{bmatrix} 1,85 \\ 4,05 \\ 4,66 \\ 5,64 \\ -0,12 \\ -0,19 \\ -0,12 \\ 0 \\ -0,94 \end{bmatrix}$$

Variables. Each x_{ij} represents the amount of unrefined oil type i to be refined to oil type j and each y_i represents the amount of unrefined oil type i to be sold in the optimal solution.

Slacks. To remind you, there are 9 constraints in the problem formulation. Constraints 1 - 4 represent the row material constraints, limited by the purchased unrefined oil. Constraints 5 - 7 represent the production balance constraints, giving the linear relationship between the content of octane in oil before and after the refining process. Constraint 8 and 9 represent the demand limitations.

All slack variables are zero, except from the slack variable from constraint 8: $s_8 = 5.093$. The slack variable represents the unused capacity due to the demand limitation for the refined oil type 3 (95 octane).

Reduced costs. Each reduced cost $\bar{c}_{ij}$ or $\bar{c}_i$ represents the change in profit if the corresponding variable is used in the solution. For instance, the reduced cost $\bar{c}_{12} = -2,02$ means that if the unrefined row oil type 1 (octane 68) is used to refine to type 2 (octane 90), the total profit would be reduced with \$2,02 pr. barrel. Obviously, the reduced costs corresponding to the basic variables in the optimal solution are zero.

Dual variables. Each dual variable π_k represents the change in the profit if the right side of the corresponding constraint is changed.

1. Row material constraints. The dual variable corresponding to the row material constraint $\pi_1 = 1,85$ means that if we purchased more unrefined oil type 1 (octane 68), we would earn \$1,85 pr. barrel.
2. Production balance constraints. The dual variable $\pi_5 = -0,12$ corresponding to the production balance constraint $-27x_{11} - 9x_{21} - 4x_{31} + 4x_{41} = 0$ means that if the content of octane received after the refining of oil to type 1 (min 95 octane) increases with 1, i.e., the refining process ends up with the refined oil containing octane 96, while we sell it at the price of the 95 octane, the objective function will decrease with \$0,12 pr. barrel.
3. Demand constraints. Finally, the dual variable $\pi_9 = -0,94$ means that we would loose \$0,94 pr. barrel of octane 85 if the lower bound for the demand for this type of oil increases.