



Strong Valid Inequalities

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The cutting plane algorithm

- Suppose $X = P \cap Z^n$ and suppose that we know a family F of valid inequalities $\pi x \leq \pi_0$, $(\pi, \pi_0) \in F$.
- Given a specific objective function one is not really interested in finding the complete convex hull, but only an approximation in the area around the optimal solution.



Separation algorithm

- Identify interesting families of valid inequalities that are cuts for particular problems, so given an $x^* \in R^n$: Is x^* in $\text{conv}(X)$?
 - yes : ok
 - no : find an inequality $\pi x \leq \pi_0$ s.t. $\forall x \in X$ but $\pi x^* > \pi_0$
- A separation algorithm can be either:
 - exact
 - heuristic



Dominance

Consider $P = \{x \in R_+^n : Ax \leq b\}$. Let $\pi x \leq \pi_0$ and $\mu x \leq \mu_0$ be two valid inequalities for P .

$$\pi x \leq \pi_0 \text{ dominates } \mu x \leq \mu_0 \Leftrightarrow$$

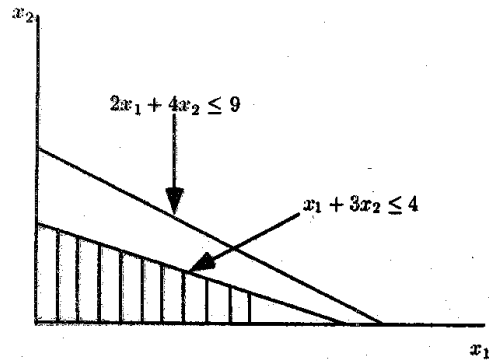
$$\exists u > 0 : u\mu \leq \pi \wedge \pi_0 \leq u\mu_0 \text{ and } (\pi, \pi_0) \neq (u\mu, u\mu_0)$$

Observe that if $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ then $\{x \in R_+^n : \pi x \leq \pi_0\} \subseteq \{x \in R_+^n : \mu x \leq \mu_0\}$.





Example



Redundance I

Let $\pi x \leq \pi_0$ be a valid inequality.

$\pi x \leq \pi_0$ is **redundant** $\Leftrightarrow$

$\exists k \geq 2$ valid ineq. $\pi^i x \leq \pi_0^i, i = 1, \dots, k$

and weights $u_i > 0, i = 1, \dots, k$

such that $(\sum_{i=1}^k u_i \pi^i) x \leq \sum_{i=1}^k u_i \pi_0^i$
dominates $\pi x \leq \pi_0$



Redundance II

Note that

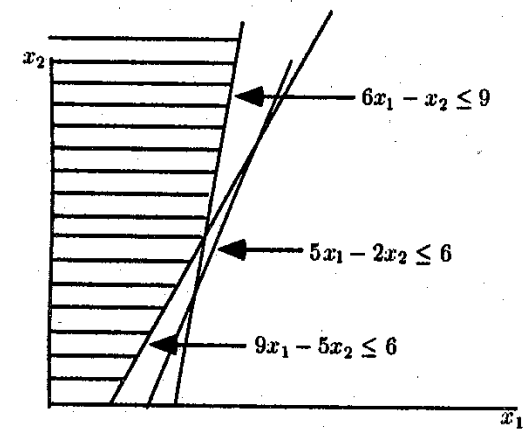
$$\{x \in R_+^n : \pi^i x \leq \pi_0^i \text{ for } i = 1, \dots, k\} \subseteq$$

$$\{x \in R_+^n : (\sum_{i=1}^k u_i \pi^i) x \leq \sum_{i=1}^k u_i \pi_0^i\} \subseteq$$

$$\{x \in R_+^n : \pi x \leq \pi_0\}$$



Example





Redundancy: theory vs. practice

- Theoretically it is important to know which inequalities are needed or nonredundant in the description of P

BUT

- in practice it is important to avoid using an inequality when one that dominates it is readily available.



Full-dimension

Consider **full-dimensional** polyhedra:

- “contains n linearly independent directions”
- “contains $n + 1$ affinely independent points”
- P full-dimensional: P has a unique minimal description $P = \{x \in R^n : a^i x \leq b^i, i = 1, \dots, n\}$.
- Affinely independent: $x_1, x_2, \dots, x_{k+1}$ affinely independent $\Leftrightarrow (x_2 - x_1, x_3 - x_1, \dots, x_{k+1} - x_1)$ as vectors are linearly independent.



Faces, Facets I

- The dimension of P , $\dim(P)$, is “one less than the maximum number of affinely independent points in P ”.
- So P full-dimensional in $R^n \Leftrightarrow \dim(P) = n$.
- F is a **face** of $P \Leftrightarrow F = \{x \in P : \pi x = \pi_0\}$ for some valid inequality $\pi x \leq \pi_0$.
- F is a **facet** of $P \Leftrightarrow F$ is a face and $\dim(F) = \dim(P) - 1$.



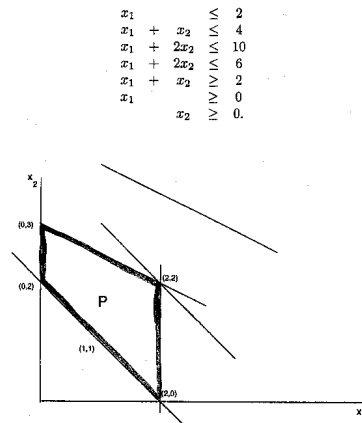
Faces, Facets II

- Key observation:** If P is full-dimensional, then a valid inequality $\pi x \leq \pi_0$ is necessary in the description of $P \Leftrightarrow \pi x \leq \pi_0$ defines a facet of P .
- Conclusion:** Facets are the strongest valid inequalities.





Example



Facets and Convex Hull

Two important questions regarding facets and convex hulls proofs:

- **Problem 1:** Given $X \subset Z_+^n$ and a valid inequality show that it defines a facet of $\text{conv}(X)$.
- **Problem 2:** Show that the polyhedron $P = \{x \in R^n : Ax \leq b\}$ describes $\text{conv}(X)$.



Problem 1: Facet-defining inequalities

Two main approaches are available:

- Use the definition: Find n points satisfying $\pi x = \pi_0$ and then prove that they are affinely independent.



- Indirect proof of affine independence:
 1. Select $t \geq n$ points $x^1, \dots, x^t \in X$ satisfying $\pi x = \pi_0$
 2. Solve $\sum_{j=1}^n \mu_j x_j^k = \mu_0 \quad k = 1, \dots, t$ in $n+1$ unknowns (μ, μ_0) .
 3. If the only solution is $(\mu, \mu_0) = \lambda(\pi, \pi_0), \lambda \neq 0$ then the inequality is facet defining.





Problem 2: Convex hull-checking

Wolsey present 8 different approaches. Let me highlight the 3 most important.

1. A or A, b has a special structure eg. totally unimodular.
2. Show that no point with fractional values can be extreme.
3. Show that for all $c \in R^n$ $z^{LP} = \max\{cx : Ax \leq b\}$ has integral solutions.



Price-collecting TSP

The travelling salesman makes a profit of f_j by visiting city j , and he pays a cost of c_e for traversing arc e . We do no longer require that all cities have to be visited. Instead we want to determine the subtour starting and ending in city 1 that maximized his profit. Variables:

- For each edge $x_e = 1$ if e is part of the subtour and $x_e = 0$ otherwise
- For each city $y_j = 1$ if city j is visited in the subtour.

This problem is as the original TSP $\mathcal{NP}$ -hard.



Price-collecting TSP – model

$$\begin{aligned}
 \max \quad & \sum_{j \in N} f_j y_j - \sum_{e \in E} c_e x_e \\
 \text{st.} \quad & \sum_{e \in \delta(i)} x_e = 2y_i \quad \text{for } i \in N \\
 & \sum_{e \in E(S)} x_e \leq \sum_{i \in S \setminus \{k\}} y_i \quad \text{for } k \in S, S \subset N \setminus \{1\} \\
 & y_1 = 1 \\
 & x \in B^{|E|}, y \in B^{|N|}
 \end{aligned}$$

How do we deal with the exponential number of generalized subtour elimination constraints.



Valid inequalities for Knapsack problems

- Consider the set $X = \{x \in B^n : \sum_{j=1}^n a_j x_j \leq b\}$. We assume that the coefficients a_j and the RHS b are positive.
- **Idea:** Identify “minimal” set of indices such that not all variables corresponding to these can be 1.
- $C \subseteq N$ is a **cover** $\Leftrightarrow \sum_{j \in C} a_j > b$
- **C Minimal cover**
 $\Leftrightarrow C$ is a cover and $\forall k \in C \sum_{j \in C \setminus \{k\}} a_j \leq b$





Example

Consider

$$7x_1 + 8x_2 + 4x_3 + 6x_4 + 9x_5 + 2x_6 \leq 16$$

Here

- $C = \{1, 2, 3\}$ is a cover and it is also minimal.
- $C = \{2, 5, 6\}$ is a cover but it is not minimal since $\{2, 5\}$ is a subset of C and is also a cover.



Cover constraints

If $C \subseteq N$ cover then

$$\sum_{j \in C} x_j \leq |C| - 1$$

is valid.



Strengthening

One might now consider whether we can

- prove that the cover inequality is strong, or
- strengthen the cover inequality to provide better cover inequalities.

First if $C \subseteq N$ cover then we can strengthen the cover inequality to

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid, where $E(C) = C \cup \{j : a_j \geq a_i \ \forall i \in C\}$.