Lagrangian Duality

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Today's Topics

- Lagrange Multipliers
- Lagrangian Relaxation
- Lagrangian Duality

Example: Economic Order Quantity Model

- Parameters
  - Demand rate: d
  - Order cost: K
  - Unit cost: c
  - Holding cost: h
- Decision Variable: The optimal order quantity Q
- Objective Function:

$$\text{minimize } \frac{dK}{Q} + dc + \frac{hQ}{2}$$

- Optimal Order Quantity:

$$Q^* = \sqrt{\frac{2dK}{h}}$$

EOQ Model

Consider the following extension

Suppose we have several items with a space constraint q

The problem

$$\begin{aligned} \text{minimize} \quad & \sum_j \left(\frac{d_j K_j}{Q_j} + d_j c_j + \frac{h Q_j}{2} \right) \\ \text{subject to:} \quad & \sum_j Q_j \leq q \end{aligned}$$

We have the following possibilities

1. The constraint is non-binding/slack, i.e

$$\sum_j \sqrt{\frac{2 d_j K_j}{h}} < q$$

2. The constraint is binding/active

Lagrange Multiplier

The problem

$$\begin{aligned} \text{minimize} \quad T(Q_1, Q_2, \ldots, Q_n) &= \sum_j \left(\frac{d_j K_j}{Q_j} + d_j c_j + \frac{hQ_j}{2} \right) \\ \text{subject to:} \quad \sum_j Q_j &= q \end{aligned}$$

- Lagrange multiplier λ

$$\text{minimize } T(Q_1, Q_2, \ldots, Q_n) + \lambda\left(\sum_j Q_j - q\right)$$

- Differentiate with respect to Q_j:

$$\frac{\partial L}{\partial Q_j} = -\frac{d_j K_j}{Q_j^2} + \frac{h}{2} + \lambda = 0 \ \forall j$$

- Solving for Q_j

$$Q_j = \sqrt{\frac{2 d_j K_j}{h + 2\lambda}} \ \forall j$$

Lagrange multiplier
Continued

- Substituting this into the constraint we have

$$\sum_j \sqrt{\frac{2d_jK_j}{h+2\lambda}} = q$$

- Solving for λ and Q_j gives:

$$\lambda = \lambda^* = \frac{\left(\frac{\sum_j \sqrt{2d_jK_j}}{q}\right)^2 - h}{2}$$

$$Q_j^* = \sqrt{\frac{2d_jK_j}{h+2\lambda^*}} \; \forall j$$

Interpretation of λ

- In linear programming a dual variable is a shadow price:

$$y_i^* = \frac{\partial Z^*}{\partial b_i}$$

- Similarly, in the EOQ model, the Lagrange multiplier measures the marginal change in the total cost resulting from a change in the available space

$$\lambda^* = \frac{\partial T^*}{\partial q}$$

Example

Problem

$$\begin{aligned} \text{minimize:} \quad & x^2 + y^2 + 2z^2 \\ \text{subject to:} \quad & 2x + 2y - 4z \geq 8 \end{aligned}$$

- The Lagrangian is:

$$L(x, y, z, \mu) = x^2 + y^2 + 2z^2 + \mu(8 - 2x - 2y + 4z)$$

- Note that the unconstrained minimum $x = y = z = 0$ is not feasible

Example Continued

- Differentiating with respect to x, y, z

$$\frac{\partial L}{\partial x} = 2x - 2\mu = 0$$

$$\frac{\partial L}{\partial y} = 2y - 2\mu = 0$$

$$\frac{\partial L}{\partial z} = 4z + 4\mu = 0$$

- We can conclude that $z = -\mu, x = y = \mu$
- Substituting this into $2x + 2y - 4z = 8$ gives $x = 1, y = 1, z = -1$
- Optimal objective function value $= 4$

Checking the value of μ

- $\mu = 1 \rightarrow$ states that we can expect an increase (decrease) of one unit for a unit change in the right hand side of the constraint
- Resolve the problem with a righthandside on the constraint of 9
- $\mu^* = \frac{9}{8}, x^* = \frac{9}{8}, y^* = \frac{9}{8}, z^* = -\frac{9}{8}$
- New objective function value:

$$\left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2 + 2\left(-\frac{9}{8}\right)^2 = \frac{324}{64}$$

- This is an increase of ≈ 1 unit

Lagrange relaxation

Problem $\mathcal{P}$

$$\begin{aligned} \text{minimize:} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \end{aligned}$$

- Choose non negative multipliers $\mathbf{u}$
- Solve the Lagrangian: $\text{minimize } f(\mathbf{x}) + \mathbf{u}\mathbf{g}(\mathbf{x})$,
- Optimal solution $\mathbf{x}^*, \mathbf{u}^*$

Lagrange relaxation
Continued

- $f(\mathbf{x}^*) + \mathbf{u}^*\mathbf{g}(\mathbf{x}^*)$ provides a lower bound $\mathcal{P}$
- If $\mathbf{g}(\mathbf{x}^*) \leq \mathbf{0}$, $\mathbf{u}^*\mathbf{g}(\mathbf{x}^*) = \mathbf{0}$, $\mathbf{x}^*$ is an optimal solution to problem $\mathcal{P}$
- $\mathbf{x}^*$ is an optimal solution to:

$$\begin{aligned} \text{minimize:} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{g}(\mathbf{x}^*) \end{aligned}$$

Class Exercises

- Proofs
- Equality Constraints

Example from last time ...

Example

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 + x_2^2 \\ \text{subject to:} \quad & x_1 + x_2 = 1 \end{aligned}$$

$$L(x_1, x_2, \lambda_1) = 2x_1^2 + x_2^2 + \lambda_1(1 - x_1 - x_2)$$

Different values of λ_1

$\lambda_1 = 0 \rightarrow$ get solution $x_1 = x_2 = 0, 1 - x_1 - x_2 = 1$

$$L(x_1, x_2, \lambda_1) = 0$$

$\lambda_1 = 1 \rightarrow$ get solution $x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, 1 - x_1 - x_2 = \frac{1}{4}$

$$L(x_1, x_2, \lambda_1) = \frac{5}{8}$$

$\lambda_1 = 2 \rightarrow$ get solution $x_1 = \frac{1}{2}, x_2 = 1, 1 - x_1 - x_2 = -\frac{1}{2}$

$$L(x_1, x_2, \lambda_1) = \frac{1}{2}$$

$\lambda_1 = \frac{4}{3} \rightarrow$ get solution $x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, 1 - x_1 - x_2 = 0$

$$L(x_1, x_2, \lambda_1) = \frac{2}{3}$$

Lagrangian Dual

Primal

$$\begin{aligned} \text{minimize:} \quad & f(\mathbf{x}) \\ \text{subject to:} \quad & \mathbf{g}(\mathbf{x}) \le \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned}$$

Lagrangian Dual

$$\begin{aligned} \text{maximize:} \quad & \theta(\mathbf{u}, \mathbf{v}) \\ \text{subject to:} \quad & \mathbf{u} \ge \mathbf{0} \end{aligned}$$

$$\theta(\mathbf{u}, \mathbf{v}) = \min_{\mathbf{x}} \{ f(\mathbf{x}) + \mathbf{u}\mathbf{g}(\mathbf{x}) + \mathbf{v}\mathbf{h}(\mathbf{x}) \}$$

Lagrangian Dual
Continued

Weak Duality: For Feasible Points

$$\theta(\mathbf{u}, \mathbf{v}) \leq f(\mathbf{x})$$

Strong Duality: Under Constraint Qualification

If f and $\mathbf{g}$ are convex and $\mathbf{h}$ is affine, the optimal objective function values are equal

- Often there is a duality gap

Example 1

The problem

$$\begin{aligned} \text{minimize:} \quad & x_1^2 + x_2^2 \\ \text{subject to:} \quad & x_1 + x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Let $X := \{x \in \mathbb{R}^2 | x_1, x_2 \geq 0\} = \mathbb{R}^2_+$
- The Lagrangian is:

$$L(\mathbf{x}, \lambda) = x_1^2 + x_2^2 + \lambda(4 - x_1 - x_2)$$

Example 1
Continued

The Lagrangian dual function:

$$\begin{aligned}\theta(\lambda) &= \min_{\mathbf{x}\in X}\{x_1^2 + x_2^2 + \lambda(4 - x_1 - x_2)\}\\ &= 4\lambda + \min_{\mathbf{x}\in X}\{x_1^2 + x_2^2 - \lambda x_1 - \lambda x_2)\}\\ &= 4\lambda + \min_{x_1\geq 0}\{x_1^2 - \lambda x_1\} + \min_{x_2\geq 0}\{x_2^2 - \lambda x_2\}\end{aligned}$$

- For a fixed value of $\lambda \geq 0$, the minimum of $L(\mathbf{x}, \lambda)$ over $x \in X$ is attained at $x_1(\lambda) = \frac{\lambda}{2}, x_2(\lambda) = \frac{\lambda}{2}$

$$\Rightarrow L(\mathbf{x}(\lambda), \lambda) = 4\lambda - \frac{\lambda^2}{2} \quad \forall \lambda \geq 0$$

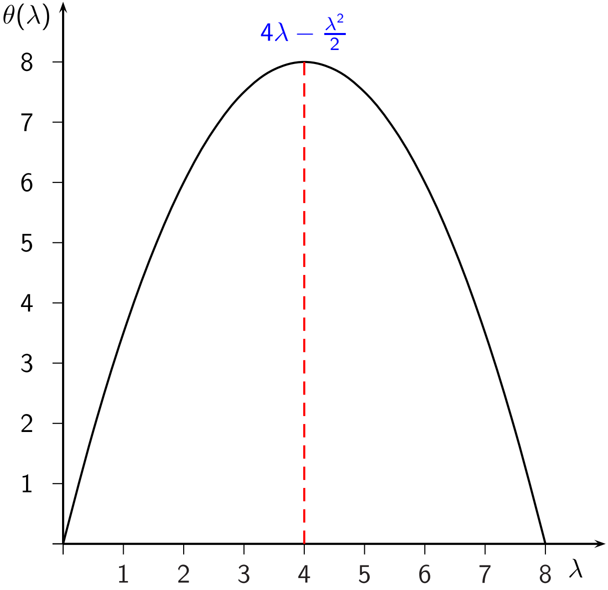
Example 1
Continued

- The dual function is concave and differentiable
- We want to maximize the value of the dual function

$$\frac{\partial L}{\partial \lambda} = 4 - \lambda = 0$$

- This implies $\lambda^* = 4, \theta(\lambda^*) = 8$
- $\mathbf{x}(\lambda^*) = \mathbf{x}^* = (2, 2)$

Example 1
Graph of Dual Function



Example 2

The problem

$$\begin{aligned} \text{minimize:} \quad & 3x_1 + 7x_2 + 10x_3 \\ \text{subject to:} \quad & x_1 + 3x_2 + 5x_3 \geq 7 \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

- Let $X := \{x \in \mathbb{R}^3 | x_j \in \{0, 1\}, j = 1, 2, 3\}$
- The Lagrangian is:

$$L(\mathbf{x}, \lambda) = 3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)$$

Example 2
Continued

The Lagrangian dual function:

$$
\begin{aligned}
\theta(\lambda) = & \min_{x \in X} \{3x_1 + 7x_2 + 10x_3 + \lambda(7 - x_1 - 3x_2 - 5x_3)\} \\
= & \; 7\lambda + \min_{x_1 \in \{0,1\}} \{(3-\lambda)x_1\} + \min_{x_2 \in \{0,1\}} \{(7-3\lambda)x_2\} \\
& \min_{x_3 \in \{0,1\}} \{(10-5\lambda)x_3\}
\end{aligned}
$$

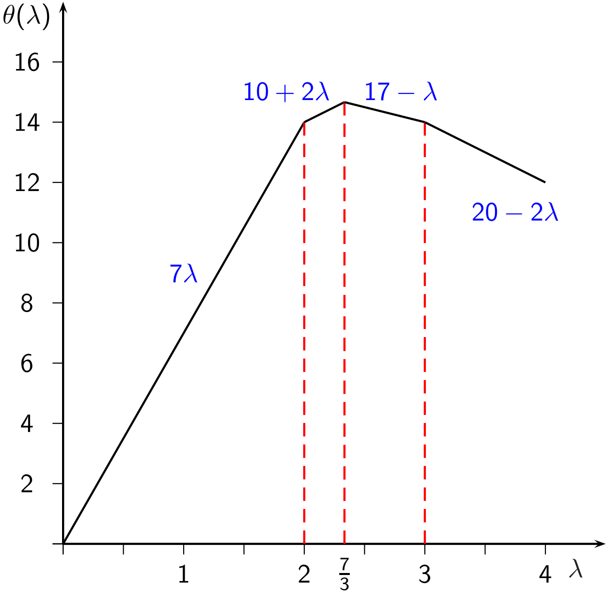
- $X(\lambda)$ is obtained by setting

$$
x_j = \begin{cases} 1 \\ 0 \end{cases} \text{when the objective coefficient is} \begin{cases} \leq 0 \\ \geq 0 \end{cases}
$$

Example 2
Lagrangian Dual Function Solution

$\lambda \in$	$x_1(\lambda)$	$x_2(\lambda)$	$x_3(\lambda)$	$g(\mathbf{x}(\lambda))$	$\theta(\lambda)$
$[-\infty, 2]$	0	0	0	7	7λ
$[2, \frac{7}{3}]$	0	0	1	2	10+2λ
$[\frac{7}{3}, 3]$	0	1	1	-1	17-λ
$[3, \infty]$	1	1	1	-2	20-2λ

Example 2
Graph of Dual Function



Example 2
Continued

- θ is concave, but non-differentiable at break points $\lambda \in \{2, \frac{7}{3}, 3\}$
- Check that the slope equals the value of the constraint function
- The slope of θ is negative for objective pieces corresponding to feasible solutions to the original problem
- The one variable function θ has a "derivative" which is non-increasing; this is a property of every concave function of one variable
- $\lambda^* = \frac{7}{3}, \theta(\lambda^*) = \frac{44}{3}$
- $\mathbf{x}^* = (0, 1, 1), f(\mathbf{x}^*) = 17$
- A positive duality gap!
- $X(\lambda^*) = \{(0, 0, 1), (0, 1, 1)\}$

Example 3
Continued

The problem

$$\begin{aligned} \text{minimize:} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \text{ free} \end{aligned}$$

- Objective:

$$f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$

- Identifying $\mathbf{g}(\mathbf{x})$

$$\mathbf{g}(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$$

- Lagrangian dual function:

$$\theta(\boldsymbol{\lambda}) = \min_{\mathbf{x}}\{\mathbf{c}^T\mathbf{x} + \boldsymbol{\lambda}^T(\mathbf{b} - A\mathbf{x})\} = \boldsymbol{\lambda}^T\mathbf{b}$$

Example 3
Continued

- Provided that the following condition is satisfied

$$A^T \boldsymbol{\lambda} - \mathbf{c} = \mathbf{0}$$

- That is, we get the following problem

$$\begin{aligned} \text{maximize:} \quad & \mathbf{b}^T \boldsymbol{\lambda} \\ \text{subject to:} \quad & A^T \boldsymbol{\lambda} = \mathbf{c} \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{aligned}$$

- Compare with Dual: min. $\mathbf{b}^T \boldsymbol{\lambda}$ s.t. $A^T \boldsymbol{\lambda} = \mathbf{c}, \boldsymbol{\lambda} \geq \mathbf{0}$

Class exercise 1

$$\begin{aligned} \text{minimize:} \quad & x \\ \text{subject to:} \quad & x^2 + y^2 = 1 \end{aligned}$$

- Solve the problem
- Formulate and solve the dual
- Check whether the objective functions are equal

Class Exercise 2

$$\begin{aligned} \text{minimize:} \quad & -2x_1 + x_2 \\ \text{subject to:} \quad & x_1 + x_2 = 3 \\ & (x_1, x_2) \in X \end{aligned}$$

1. Suppose X={(0,0),(0,4),(4,4),(4,0),(1,2),(2,1)}
2. Formulate the Lagrangian Dual Problem
3. Plot the Lagrangian Dual Problem
4. Find the optimal solution to the primal and dual problems
5. Check whether the objective functions are equal
6. Explain your observation in 5