Exercise DO.1: Down payment of a loan

Static and Dynamic Optimization

Background

Matlab files (i.e. m-files) for this exercise (including solutions) are available on the course home page.

The two first exercises are meant as primers and can be passed. You can with a minimal effort run the solutions runex1.m and runex2.m.

In connection to dynamic optimization we are often faced with a two point boundary problem, where state conditions are given in the start and costate conditions are given in the end of the period. One way to solve this problem is to use a shooting method, in which the costate recursion is reversed, such that it is a recursion forward in time. In that situation we are forced to search for the correct initial value of the costate such that the end point condition is met. In that search the following two procedures are useful.

1 Optimization

Try to minimize

$$J = \|u - \begin{bmatrix} 1 \\ 3 \end{bmatrix}\|^2 + 1$$

wrt. the two dimensional vector u using eg. fminsearch. Write a short m-file (e.g. with the name: fopt.m) with the following contents

```
function loss=fopt(u)
loss=sum((u-[1;3]).^2)+1;
```

and execute the matlab commands (or put them into a script)

```
opt=optimset;
opt=optimset(opt,'Display','iter');
fminsearch('fopt',[1; 1],opt)
```

The 2 first lines are just included for controlling the output (to display) from fminsearch.

2 Solving a set of equation

Try to use the matlab command fsolve to find the solution to the problem ($u \in \mathbb{R}^2$):

$$u - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

Write a short m-file (e.g. with the name: fz.m) with the following contents

```
function f=fz(u)
f=u-[1;3];
```

and execute the matlab commands

```
opt=optimset;
opt=optimset(opt,'Display','off');
fsolve('fz',[1; 1],opt)
```

3 Dynamic Optimization

Consider the problem of payment of a (study) loan which at the start of the period is 50.000 Dkr. Let us focus on the problem for a period of 10 years. We are going to determine the optimal pay back strategy for this loan, i.e. to determine how much we have to pay each year. Assume that the rate of interests is 5 % per year ($\alpha = 0.05$) (and that the loan is credited each year), then the dynamics of the problem can be described by:

$$x_{i+1} = ax_i + bu_i \qquad a = 1 + \alpha \qquad b = -1$$

where x_i is the actual size of the loan (including interests) and u_i is the annual payment.

On one hand, we are interested in minimizing the amount we have to pay to the bank. On the other hand, we are not interested in paying to much each year. The objective function, which we might use in the minimization could be

$$J = \frac{1}{2}px_N^2 + \sum_{i=0}^{N-1} \frac{1}{2}qx_i^2 + \frac{1}{2}ru_i^2$$

where $q = \alpha^2$. The weights r and p are at our disposal. Let us for a start choose $r = q$ and $p = q$ (but let the parameters be variable in your program in order to change them easily).

Question: 1 As a start identify the quantities (N, x_0, f, ϕ and L) that specifies the problem. □

Question: 2 Write down the Hamiltonian function. □

Question: 3 Find the derivatives that enter into the Euler-Lagrange equation. □

Question: 4 Write down the Euler-Lagrange equations (KKT conditions) for this problem and verify they are:

$$\begin{aligned} x_{i+1} &= ax_i + bu_i \qquad x_0 = \underline{x}_0 \\ \lambda_i &= qx_i + a\lambda_{i+1} \qquad \lambda_N = px_N \\ 0 &= ru_i + b\lambda_{i+1} \end{aligned}$$

□

Question: 5 Solve the stationarity condition with respect to u_i, i.e. express u_i as function of λ_{i+1}. □

Question: 6 Reverse the recursion for the costate, i.e. express λ_{i+1} as function of x_i and λ_i. □

Question: 7 Assume now, that both the initial x_0 and λ_0 are given. Verify (check, convince yourself or simply accept) that the Euler-Lagrange equations are equivalent to

$$\begin{aligned} \lambda_{i+1} &= \frac{\lambda_i - qx_i}{a} \\ u_i &= -\frac{b}{r}\lambda_{i+1} \\ x_{i+1} &= ax_i + bu_i \end{aligned} \tag{1}$$

which can be solved for $i = 0,\ 1,\ \ldots\ N-1$. □

The problem in the method described above is that the initial value λ_0 is not known. However the costate at the end is known to obey (the end point constraint):

$$\lambda_N = px_N \qquad \text{or} \qquad \lambda_N - px_N = 0$$

One method is to guess λ_0 and check if the end point condition on the costate is satisfied. Notice, x_0 is known (given by the problem).

The solution to the following question can be found in fejlf.m.

Question: 8 Write a piece of matlab code that solves the recursions in (1). The input to the function is a guess on the initial costate (ie. λ_0) and the output is the error between the final costate and is correct value. □

The solution to the next two questions can be found in runex3.m.

Question: 9 Use eg. fsolve (in matlab) to find the correct initial costate value. □

Question: 10 plot the variation of x_i and u_i. Study the effect of the parameters p, r and q by changing their values. Try eg. $r = 10q$ and $r = 0.1q$ and $p = 0$ and $p = 100 * q$.
□