Linear Programming & Duality

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Today’s Agenda

- Linear Programming
- Revised Simplex
- Duality
Linear Programming
Solution Method
Linear Programming

Simplex Method

- Convert $\leq$ inequalities by adding slack variables
- Put data into simplex tableau
- Perform simplex iterations by pivoting
- **Entering Variable** (*pivot column*)
  - Most negative coefficient in top row
- **Leaving Variable** (*pivot row*)
  - Minimum ratio: right hand sides and positive pivot column entries
- We disregard complications here
  - Phase 1, no feasible solution, unbounded solutions
## Linear Programming

### First and Final Tableau

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-3$</td>
<td>$-5$</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3/2</td>
<td>1</td>
<td>1</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1/3</td>
<td>$-1/3$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td></td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$-1/3$</td>
<td>$1/3$</td>
<td></td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
Revised Simplex Matrix formulation

• General LP

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to:} & \quad Ax & \leq b \\
& \quad x & \geq 0
\end{align*}
\]

• becomes ..

\[
\begin{align*}
\text{maximize} & \quad c^T x + 0s \\
\text{subject to:} & \quad Ax + Is & = b \\
& \quad x & \geq 0 \\
& \quad s & \geq 0
\end{align*}
\]
• In each tableau each variable in $x, s$ is designated as a basic variable or a nonbasic variable

• The tableau represents the equation system solved with respect to the basic variables.

• The basis matrix $B$ is formed by the columns in the first tableau of the current basic variables

• The inverse basis matrix appears under the slack variables in each tableau
Revised Simplex

Matrix formulation

maximize \quad c^T_B x_B + c^T_N x_N

subject to: \quad B x_B + N x_N = b
\quad x_B \quad \geq 0
\quad x_N \quad \geq 0
Revised Simplex

First and Later Tableau

First tableau ...

\[
\begin{array}{ccc|c|c}
  Z & x & s & & \\
  1 & -c & 0 & 0 & \\
  A & I & b & & \\
\end{array}
\]

Later tableau ...

\[
\begin{array}{ccc|c|c}
  Z & x & s & & \\
  1 & c_B B^{-1} A - c & c_B B^{-1} & c_B B^{-1} b & \\
  B^{-1} A & B^{-1} & B^{-1} b & & \\
\end{array}
\]

- The current solution is \( x_B = B^{-1} b, \ x_N = 0, \ Z = c_B B^{-1} b \)
- At Optimality we have \( c_B B^{-1} \geq 0, \ c_B B^{-1} A \geq c \)
- The shadow prices are \( c_B B^{-1} \)
• The *primal* problem

\[
\begin{align*}
\text{maximize} & \quad Z_P = 3x_1 + 5x_2 \\
\text{subject to:} & \quad x_1 \leq 4 \\
& \quad +2x_2 \leq 12 \\
& \quad 3x_1 + 2x_2 \leq 18 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \geq 0
\end{align*}
\]
The corresponding *dual* problem

\[
\begin{align*}
\text{minimize} & \quad Z_D = 4y_1 + 12y_2 + 18y_3 \\
\text{subject to:} & \quad Z_D = y_1 + 3y_3 \geq 3 \\
& \quad + 2y_2 + 2y_3 \geq 5 \\
& \quad y_1 \geq 0 \\
& \quad y_2 \geq 0 \\
& \quad y_3 \geq 0
\end{align*}
\]
The problem
Each unit of product 1 requires 1 hour in department A and 1 hour in department B, and yields a profit of 1. The corresponding numbers for product 2 are 1 and 3, and 2. There are 3 and 7 hours available in departments A and B, respectively.

- Formulate an LP model and set up the first tableau
- Write the dual problem
Duality

Weak duality theorem

Primal: maximize $c^T x$

subject to: $Ax \leq b$

$x \geq 0$

Dual: minimize $b^T y$

subject to: $A^T y \geq c$

$y \geq 0$

Weak Duality Theorem

If $x$ is primal feasible and $y$ is dual feasible, then $c^T x \leq y^T Ax \leq b^T y$

Proof?
Strong duality theorem

**Strong Duality Theorem**

If one of the problems has an optimal solution the other one also has an optimal solution and the optimal objective function values are equal

- The optimal dual solution appears in the optimal primal tableau, under the slack variables (Proof?)
- The two other possibilities are
  - One problem is infeasible, the other is unbounded
  - Both problems are infeasible
Duality

Complementary Slackness

Primal: maximize \( c^T x \)

subject to: \( A x + s = b \)
\( x \geq 0 \)
\( s \geq 0 \)

Dual: minimize \( b^T y \)

subject to: \( A^T y - e = c \)
\( y \geq 0 \)
\( e \geq 0 \)

Definition

A primal solution and a dual solution exhibit complementary slackness if \( e^T x = 0 \) and \( y^T s = 0 \), i.e., corresponding \( x \)- and \( y \)-values are not both positive.
Complementary Slackness

Complementary Slackness Theorem
Theorem: A primal solution and a dual solution are optimal iff they are feasible and complementary (proof?)

Example
Correspondences: \(x_1\) and \(e_1\), \(x_2\) and \(e_2\), \(y_1\) and \(s_1\), \(y_2\) and \(s_2\), \(y_3\) and \(s_3\)
The final tableau for the exercise 1 problem is

\[
\begin{array}{c|cccc|c}
Z & x_1 & x_2 & s_1 & s_2 \\
1 & 1 & 3/2 & 1/2 & 2/2 & 5/2 \\
1 & 3/2 & -1/2 & 1/2 & 2/2 & 1/2 \\
1 & -1/2 & 1/2 & 2/2 & 1/2 & 2/2 \\
\end{array}
\]

- Read off the optimal solution and the dual solution.
- Read off \( B^{-1} \) and verify that \( B^{-1}B = I \).
- Given the primal solution, find the dual solution using complementary slackness.
- Use complementary slackness to show that \( x_1 = 0, \ x_2 = \frac{7}{3} \) is not optimal.