Karush-Kuhn-Tucker Conditions

Inequality constraints
Constraint qualification
Equality constraints

Henrik Juel, DTU Management

Inequality constraints

min. $f(\mathbf{x})$ s.t. $g_i(\mathbf{x}) \leq 0 \,\forall\, i$

Feasible solution $\mathbf{x}^o$
Binding constraints $I = \{i : g_i(\mathbf{x}^o) = 0\}$
Constraint qualification

KKT necessary optimality conditions:
If $\mathbf{x}^o$ is a local minimum,
then there exist multipliers $u_i \geq 0$ for $i \in I$
such that $f'(\mathbf{x}^o) + \sum_{i \in I} u_i g_i'(\mathbf{x}^o) = \mathbf{0}$

Inequality example

min. $(x-3)^2 + (y-2)^2$
s.t. $x^2 + y^2 \leq 5,\ x + 2y \leq 4$
and $x, y \geq 0$

The gradients are: $f' = (2x-6, 2y-4)$
$g_1' = (2x, 2y),\ g_2' = (1, 2)$
$g_3' = (-1, 0),\ g_4' = (0, -1)$

Consider the point $(x, y) = (2, 1)$
It is feasible with $I = \{1, 2\}$

$(-2, -2) + u_1(4, 2) + u_2(1, 2) = (0, 0)$
$u_1 = 1/3,\ u_2 = 2/3$

Class exercise: Solve max. $\ln(x+1) + y$
s.t. $2x + y \leq 3,\ x, y \geq 0$

Geometric interpretation

$f'(\mathbf{x})$ points in the direction of steepest ascent
$-f'(\mathbf{x})$ points in the direction of steepest descent

In two dimensions:
$f'(\mathbf{x}^o)$ is perpendicular to a level curve of f
$g_i'(\mathbf{x}^o)$ is perpendicular to the level curve $g_i(x, y) = 0$

Constraint qualification example:
max. x s.t. $y \leq (1-x)^3,\ y \geq 0$

Consider the global max: $(x, y) = (1, 0)$ at a cusp
After reformulation, the gradients are: $f' = (-1, 0)$
$g_1' = (3(x-1)^2, 1) = (0, 1),\ g_2' = (0, -1)$

No u_1, u_2 exist with
$(-1, 0) + u_1(0, 1) + u_2(0, -1) = (0, 0)$

Constraint qualifications

KKT constraint qualification:
$g_i'(\mathbf{x}^o)$ for $i \in I$ are linearly independent

Slater constraint qualification:
$g_i(\mathbf{x})$ for $i \in I$ are convex functions
A nonboundary point exists: $g_i(\mathbf{x}) < 0$ for $i \in I$

Class exercise: Solve
min. $x^2 + y^2$
s.t. $x^2 + y^2 \leq 5,\ x + 2y = 4$
and $x, y \geq 0$

Sufficient condition

min. $f(\mathbf{x})$ s.t. $g_i(\mathbf{x}) \leq 0 \,\forall\, i$

If f and g_i for $i \in I$ are convex functions,
then a feasible KKT point is optimal

An equality constraint is equivalent to two inequality constraints:
$h(\mathbf{x}) = 0$
$h(\mathbf{x}) \leq 0$ and $-h(\mathbf{x}) \leq 0$

The corresponding two nonnegative multipliers may be combined to one free one:
$u_+ h'(\mathbf{x}) + u_-(-h'(\mathbf{x})) = v h'(\mathbf{x})$

Equality constraints also

min. $f(\mathbf{x})$ s.t. $g_i(\mathbf{x}) \leq 0 \,\forall\, i,\ h_j(\mathbf{x}) = 0 \,\forall\, j$

As before, let $\mathbf{x}^o$ be a feasible solution, define
$I = \{i : g_i(\mathbf{x}^o) = 0\}$,
and assume that a constraint qualification holds.

Necessary optimality condition:
If $\mathbf{x}^o$ is a local minimum,
then there exist multipliers $u_i \geq 0$ for $i \in I$ and $v_j \,\forall\, j$
such that $f'(\mathbf{x}^o) + \sum_{i \in I} u_i g_i'(\mathbf{x}^o) + \sum_j v_j h_j'(\mathbf{x}^o) = \mathbf{0}$

Sufficient optimality condition:
If f and g_i for $i \in I$ are convex functions,
and $h_j \,\forall\, j$ are affine (linear),
then a feasible KKT point is optimal

Alternative formulation

min. $f(\mathbf{x})$ s.t. $g_i(\mathbf{x}) \leq 0 \,\forall\, i,\ h_j(\mathbf{x}) = 0 \,\forall\, j$

This form of KKT condition is more common:

$f'(\mathbf{x}^o) + \sum_i u_i g_i'(\mathbf{x}^o) + \sum_j v_j h_j'(\mathbf{x}^o) = \mathbf{0}$
$u_i g_i(\mathbf{x}^o) = 0,\ u_i \geq 0 \,\forall\, i$
$\mathbf{x}^o$ feasible

Or i vector form:
min. $f(\mathbf{x})$ s.t. $\mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}$

$f'(\mathbf{x}^o) + \mathbf{u}\mathbf{g}'(\mathbf{x}^o) + \mathbf{v}\mathbf{h}'(\mathbf{x}^o) = \mathbf{0}$
$\mathbf{u}\mathbf{g}(\mathbf{x}^o) = 0,\ \mathbf{u} \geq \mathbf{0}$
$\mathbf{x}^o$ feasible

Class exercise: Write the KKT conditions for
max. cx s.t. $Ax \leq b,\ x \geq 0$