Course 02647, Fall 2003
Applied Mathematics for Physicists

HOME WORK 6

Calculus of Variations

1. Write down the Euler equations for the functions $u(x, y)$ and $v(x, y)$ for which the functional

$$J = \iint_{(x_1,y_1)}^{(x_2,y_2)} f(u, u_x, u_{yy}, u_{xxx}, v, v_y, v_{xx}, v_{yyy}, x, y)\, dxdy, \tag{1}$$

has an extreme value (give the requirements on the variations $\eta_u(x, y)$ and $\eta_v(x, y)$). Here and in the following a subscript denotes differentiation, i.e. $u_{yy} = \partial^2 u/\partial y^2$.

2. Consider the functional

$$J = \int_{x_1}^{x_2} \sqrt{1 + y_x^2}\, dx, \tag{2}$$

which gives the distance between two points in the xy-plane (see example 17.2.1 in the course book).

2.a) Assume as usual a variational problem in which the deformed path is described as $y = y(x, \alpha) = y(x, 0) + \alpha\eta(x)$. Expand $J(\alpha)=J[y(x, \alpha), y_x(x, \alpha), x]=J_0 + \alpha J_1 + \alpha^2 J_2$ in powers of α, and find the α-independent coefficients J_0, J_1, and J_2.

2.a) Show that the Euler equation $J_1=0$ leads to the equation for a straight line.

2.c) Show that the straight line gives the *minimum* distance.

3. Hamilton's Principle says that a physical system will evolve in time such as to give a stationary value of the integral

$$J = \int_{t_1}^{t_2} L\, dt, \tag{3}$$

of the Langrangian L. Here we consider the Lagrangian $L=L(u, u^*, u_t, u_t^*, u_x, u_x^*)$

$$L = \int_{-\infty}^{\infty} \mathcal{L} dx, \quad \mathcal{L} = \mathrm{Im}\{u^* u_t\} + |u_x|^2 - \frac{1}{2}|u|^4, \tag{4}$$

where $\mathcal{L}=\mathcal{L}(u, u^*, u_t, u_t^*, u_x, u_x^*)$ is the Lagrangian density and u^* is the complex conjugate of u.

3.a) Use Hamilton's principle to derive the dynamical equation for $u(x, t)$ (remember that u and u^* are independent - see also example 17.7.3 in the book). In other words, write down the Euler-Lagrange equations for the integral

$$J = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \mathcal{L}\, dtdx. \tag{5}$$

3.b) Consider the following guess for a stationary solution to the equation derived in question 3.a

$$u(x,t) = \alpha \exp(-\beta x^2)\exp(i\lambda t), \tag{6}$$

where the amplitude α and width β are unknown parameters. Insert the solution (6) into Eq. (4) and find the reduced Lagrangian $\tilde{L} = \tilde{L}(\alpha, \beta)$, which only depends on two independent variables α and β.

3.c) Write down the Euler-Lagrange equations for the reduced Lagrangian $\tilde{L}(\alpha, \beta)$, i.e. for the integral

$$J = \int_{t_1}^{t_2} \tilde{L}(\alpha, \beta)\, dt. \tag{7}$$

Solve the Euler-Lagrange equations and find α and β in terms of the parameter λ.

This type of variational method is also called the *Method of Collective Coordinates*, since the reduced set of parameters α and β collectively gives all necessary information about the solution, once the trial function is given [a Gaussian in Eq. (6)]. The accuracy of the method depends on the choice of trial function, as we shall see.

3.d) Check the solution you find by inserting it into the equations found in question 3.a.

3.e) Repeat questions 3.b-d with the new solution guess

$$u(x,t) = \alpha\, \mathrm{sech}(\beta x)\exp(i\lambda t), \tag{8}$$

in which a sech-profile is chosen as a trial function. Discuss the result.

4. Consider the eigenvalue problem

$$u_{xx} + \lambda u = 0, \qquad u(0) = 0, \quad u_x(1) + u(1) = 0. \tag{9}$$

Find the constant α that makes the function

$$u(x) = x^2 + \alpha x, \tag{10}$$

suitable as a trial eigenfunction. Use the Rayleigh-Ritz variational technique to find an approximate value for the ground-state (or minimum) eigenvalue. Compare it to the exact value, which is the lowest positive root of the transcendental equation $\tan(k) = -k$, where $\lambda = k^2$.