Course 02647, Fall 2003
Applied Mathematics for Physicists

HOME WORK 5

Tensor Calculus and Group Theory

1 The Levi-Cività symbol.

a) Show that

$$\sum_{lmn} A_{il} A_{jm} A_{kn} \epsilon_{lmn} = \epsilon_{ijk} det A, \tag{1}$$

where A_{ij} is an arbitrary 3×3 matrix and ϵ_{ijk} is the Levi-Cività antisymmetric symbol given by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{when } ijk \text{ is a permutation of } 123 \\ -1 & \text{when } ijk \text{ is a permutation of } 321 \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

2 Tensor of rank 2.

a) Under the coordinate rotation given by M_{ij}, the coordinates of a second rank tensor, T_{ij}, transform as

$$\bar{T}_{ij} = \sum_{kl} M_{ik} M_{jl} T_{kl}. \tag{3}$$

Show that this transformation law can be written as

$$\bar{T} = M T M^{-1} \tag{4}$$

in matrix notation.

3 Contraction.

a) Setting two indices, one covariant and the other contravariant, equal and using the summation convention is called contraction.
Show that a second-rank mixed tensor, B^i_j, becomes invariant under contraction and therefore a scalar.

4 Pauli spin matrices

$$\sigma_1 = \left\{ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right\}, \quad \sigma_2 = \left\{ \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right\}, \quad \sigma_3 = \left\{ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right\}. \tag{5}$$

In Eqs. (5), (8), and (10) i denotes the imaginary unit (not to be confused with index i).

a) Show that $\sigma_i^2 = I$, where the unit matrix I is defined as

$$I = \left\{ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\}. \tag{6}$$

b) Show that $\sigma_i \sigma_j = i\sigma_k$, where (i,j,k)=(1,2,3), (2,3,1), (3,1,2), i.e. any cyclic permutation of (1,2,3).

c) Show that $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} I$, where the Kronecker delta δ_{ij} is defined as

$$\delta_{ij} = \left\{ \begin{array}{cc} 1 & i = j \\ 0 & i \neq j \end{array} \right. . \tag{7}$$

d) Using the Pauli matrices show that

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \;\; I + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}), \tag{8}$$

where

$$\boldsymbol{\sigma} = \hat{\mathbf{x}}\sigma_1 + \hat{\mathbf{y}}\sigma_2 + \hat{\mathbf{z}}\sigma_3, \tag{9}$$

$\mathbf{a}$ and $\mathbf{b}$ are ordinary vectors, and I is given by (6).

5 Structrural constants in the Lie algebra for SO(3) and SU(2).

a) Show that the SO(3) generators

$$S_x = \left\{ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{array} \right\}, \quad S_y = \left\{ \begin{array}{ccc} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{array} \right\}, \quad S_z = \left\{ \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\} \tag{10}$$

satisfy the commutation relation

$$[S_j, S_k] = c_{jk}^l S_l, \tag{11}$$

with the structural constants $c_{jk}^l = i\epsilon_{jkl}$, ϵ_{jkl} being the Levi-Cività symbol.

b) Show that the generators in the SU(2) group, σ_1, σ_2, and σ_3, satisfy the same commutation relation with $S_j = \sigma_j/2$.

6 Dirac matrices.

a) We consider the Dirac matrix $M = \phi_0 I + \Phi$, where $\Phi = \phi_1\sigma_1 + \phi_2\sigma_2 + \phi_3\sigma_3$. Show that $tr\, M$=0 if and only if $\phi_0 = 0$.

b) Show that $\Phi^2 = |\Phi|^2 I$ and $\Phi^3 = |\Phi|^2\Phi$, where $|\Phi| = \sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2}$.

c) Show that $\sin\Phi = (\sin|\Phi|/|\Phi|)\Phi$, $\cos\Phi = \cos|\Phi| I$, and $[\sin\Phi]^2 + [\cos\Phi]^2 = I$.

d) Under what conditions does the following relation hold:

$$\frac{d}{dt}\sin\Phi = \cos\Phi \frac{d\Phi}{dt}? \tag{12}$$

Afleveringsfrist: 18/11-2003.