Course 02647, Fall 2003
Applied Mathematics for Physicists

HOME WORK 4

Separation of variables, Indicial equation, Bessel functions, Green's function

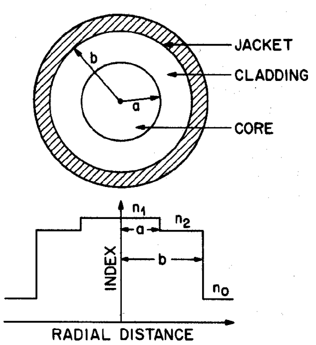

Figure 1: *Cross section and refractive index profile of a step-index optical fiber.*

The transverse profile $E(x,y)$ of the electric field propagating in an optical fiber obeys Helmholtz's equation in two dimensions

$$\nabla^2 E + (n^2k^2 - \beta^2)E = 0. \tag{1}$$

Here k is the wavenumber in vacuum and β is the longitudinal propagation constant (along z). The refractive index n measures the material density and determines how much the light is slowed down in the fiber in comparison with the speed of light in vacuum.

We consider the conventional radially symmetric step-index fiber shown in Fig. (1), for which the refractive index is a constant n_1 in the core ($r<a$) and n_2 in the cladding ($a<r<b$), where $n_2 < n_1$. Since the fiber is radially symmetric we can use polar coordinates. In the mathematical treatment of the problem we assume that the cladding extends to infinity, $b\to\infty$, which is well-justified for sufficiently bounded (localized) modes.

1.1. Express Eq. (1) in polar coordinates (r,θ) and split it into two equations, one for the radial part $R(r)$ (with $n=n_1$ for $0\leq r\leq a$ and $n=n_2$ for $r\geq a$) and one for the angular part $\Phi(\theta)$ ($0\leq\theta\leq 2\pi$), using separation of variables.

We are looking for guided (β real) bounded modes that are everywhere finite and decay monotonically to zero in the cladding. Furthermore, the electric field is a rotationally symmetric and continuous function, i.e., $\Phi(0)=\Phi(2\pi)$, $\Phi'(0)=\Phi'(2\pi)$

1.2. Consider the equation for the radial part, which has one singular point in the *finite r-plane.* Write down the indicial equation and solve it at that singular point.

In general one must analyse equations with singularities in detail using polynomial expansions, in order to find appropriate solutions. However, here the radial equation is the well-known Bessel's equation, so we take the known solutions from e.g. Schaum's Mathematical Handbook.

1.3. Solve the separated equations and write down the full solution $E(r,\theta)$ in the core and cladding. Define the transverse propagation constant in the core as $\kappa=\sqrt{n_1^2k^2-\beta^2}$ and the transverse decay rate in the cladding as $\gamma=\sqrt{\beta^2-n_2^2k^2}$. What are the bounds on the longitudinal propagation constant β?

We will now consider a vibrating string (or forced linear oscillator) described by the inhomogeneous linear partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = f(x,t), \qquad x \in [0,\pi], \quad t \geq 0, \tag{2}$$

where $u{=}u(x,t)$ is the amplitude of the vibration.

2.1) The stationary states $u(x,t){=}u_0(x)$ of the vibrating string are thus described by the inhomogeneous linear ordinary differential equation

$$\hat{L}_0 u_0(x) = u_0''(x) = f_0(x), \qquad x \in [0,\pi], \tag{3}$$

where prime denotes differentiation with respect to x, i.e., $u_0'(x) = du_0/dx$. Solve Eq. (3) for functions $u_0{=}u_0(x)$ belonging to the domain

$$D_{u_0}: \quad u_0(x) \in C^2, \quad u(0) = 1, \quad u(\pi) = -1,$$

using Greens function $G(x,x')$, keeping in mind that the boundary conditions are inhomogeneous.

2.2) Show that the Green's function $G(x,x')$ is symmetric in its arguments. Which property of the operator $\hat{L}_0$ is a consequence of the fact that the Green's function is real and symmetric (and satisfies homogeneous boundary conditions)?

2.3) Taking the Fourier transformation of Eq. (2) one obtains the following equation

$$\hat{L}\tilde{u} = \tilde{u}'' + \omega^2 \tilde{u} = \tilde{f}, \qquad x \in [0,\pi], \tag{4}$$

for the Fourier components $\tilde{u}{=}\tilde{u}(x){=}\tilde{u}(x;\omega)$, where $\tilde{u}$ and $\tilde{f}{=}\tilde{f}(x){=}\tilde{f}(x;\omega)$ depend on the frequency ω. Consider now $\omega{>}0$ and find Green's function for the problem (4) with $\tilde{u}(x)$ belonging to the domain

$$D_{\tilde{u}}: \quad \tilde{u}(x) \in C^2, \quad \tilde{u}(0) = \tilde{u}(\pi) = 0,$$

with simple homogeneous boundary conditions.

2.4) Calculate those values of ω for which $G(x,x')$ becomes unbounded. What kind of oscillations do these values of ω correspond to?

Afleveringsfrist: 4/11-2003.