Course 02647, Fall 2003
Applied Mathematics for Physicists

HOME WORK 3

Sturm-Liouville theory

1 **a)** Find the values of the constants α and β, for which the function $y(x)=x^\alpha \exp(\beta x)$ satisfies the equation

$$(x^2y')' + \lambda x^2 y = 0, \tag{1}$$

where λ is a constant and prime denotes differentiation with respect to x. Find the complete solution for $\lambda \neq 0$ and for λ=0.

b) Find all eigenvalues and eigenfunctions for the eigenvalue problem

$$(x^2y')' + \lambda x^2 y = 0, \qquad x \in [1,2], \qquad y(1) = y(2) = 0, \tag{2}$$

2 The spherically symmetric eigenoscillations in an acoustic resonator, formed by the empty space between two concentric spheres of radius a and $a+d$, respectively, are determined by the eigenvalue problem

$$(x^2y')' + \lambda x^2 y = 0, \qquad x \in [a, a+d], \qquad y'(a) = y'(a+d) = 0, \tag{3}$$

where x is the distance from the center.

a) Show that all eigenvalues are the squares of the roots of the transcendental equation

$$\tan(kd) = \frac{kd}{1 + a(a+d)k^2}, \tag{4}$$

and find the eigenfunctions belonging to the eigenvalue $\lambda_n = k_n^2$

b) Explain why all roots of the transcendental equation (4) are real and sketch a figure, from which the roots can be determined graphically.

c) Show that when $a \to \infty$ and d=1, then the eigenvalues and eigenfunctions asymptotically tend towards $n^2\pi^2$ and $c_c \cos(n\pi[x-a])$, respectively, where n=0,1,2,...

3 Consider the differential operator

$$L = f\frac{d^2}{dx^2} + g\frac{d}{dx} + h, \tag{5}$$

where f=$f(x)$, g=$g(x)$, and h=$h(x)$ are real functions of x, with $f(x) \neq 0$ for all $x \in [0,1]$. The operator L acts on functions y=$y(x)$, for which $y(0)$=$y(1)$=0. Find the adjoint operator $\tilde{L}$ and the domain $D_{\tilde{L}}$ in which it is defined.

4 Consider the eigenvalue problem

$$y'' + \lambda y = 0, \qquad x \in [0,1], \qquad y'(0) = 0, \qquad y'(1) + y(1) = 0, \tag{6}$$

which has the eigenvalues $\lambda_n = k_n^2$, where k_n are the positive roots of the equation $\cot(k) = k$. The eigenfunction belonging to λ_n is given by $\phi_n(x) = \cos(k_n x)$.

a) Find the expansion of the functions 1 and $x^2 - 3$ in terms of the set of eigenfunctions ϕ_1, ϕ_2, ...

b) Find the asymptotic order of magnitude of the expansion coefficients for $n \to \infty$.

$\boxed{\textbf{For plotting you may use any software}}$

$\boxed{\textbf{Afleveringsfrist: 21/10-2003.}}$