Course 02647, Fall 2002
Applied Mathematics for Physicists

HOME WORK 2

Impulse Response, Laplace and Fourier Transformation

1. Background on Impulse Response

In many technical problems we must find the reaction of a system (laboratory experiment, dynamical system, a chemical reaction or other) to external influences. When you will find the reaction - or better *the response* - the mathematical expression for the influence (the input) is substituted into the mathematical model of the system, which then is solved with the appropriate initial and/or boundary conditions. When the responses of many different influences are wanted, then you must solve several inhomogeneous initial or boundary value problems for the same differential operator. If we take a formal approach to the problem, then we must find the solution $y(t)$ from the operator equation

$$\hat{L}y(t) = f(t).$$

It can formally be solved in terms of the inverse operator $\hat{L}^{-1}$,

$$y(t) = \hat{L}^{-1}f(t),$$

where $\hat{L}^{-1}$ is independent of the function f. If we once and for all could evaluate $\hat{L}^{-1}$ it would be possible to reduce the solution of all initial and boundary value problems for different functions $f(t)$ to first a computation of the inverse operator $\hat{L}^{-1}$ and thereafter use $\hat{L}^{-1}$ for computations of $\hat{L}^{-1}f(t)$. That is exactly what we now are going to do by applying the principle of superposition, which holds *under the condition that the operator $\hat{L}$ is linear*.

Let us recall that we can express an integrable function $v(t)$ in the following form:

$$v(t) = \int_{-\infty}^{\infty} v(s)\delta(t-s)ds, \tag{1}$$

using the definition of the δ-function [see (1.182) page 88 in "Mathematical Methods for Physicists"]. Now we let $x(t)$ be the solution of the linear initial value problem

$$\hat{L}x(t) = \delta(t), \tag{2}$$

where $\hat{L}$ is a linear differential operator and t is time (independent variable). Because of linearity we may shift the time axis by a constant s and multiply by a function $v(s)$, which depends only on the parameter s, to obtain

$$\hat{L}\{v(s)x(t-s)\} = \delta(t-s)v(s). \tag{3}$$

Integrating over the parameter s this expression becomes

$$\int_{-\infty}^{\infty} \hat{L}\{v(s)x(t-s)\}ds = \int_{-\infty}^{\infty} v(s)\delta(t-s)ds. \tag{4}$$

Using the linearity of the operator $\hat{L}$ and the expansion (1) we obtain the equation

$$\hat{L}\int_{-\infty}^{\infty} v(s)x(t-s)ds = v(t), \tag{5}$$

which expresses that whenever $x(t)$ is the solution of the linear initial value problem (2) then $y(t) = \int_{-\infty}^{\infty} v(s)x(t-s)ds$ is the solution of the linear initial value problem

$$\hat{L}y(t) = v(t). \tag{6}$$

The conclusion is that when we have found the solution $x(t)$ of $\hat{L}x(t)=\delta(t)$ then we can calculate the solution $y(t)$ of $\hat{L}y(t)=v(t)$ *for any integrable function v(t)* by a simple integration, namely by calculating the convolution integral $\int_{-\infty}^{\infty} v(s)x(t-s)ds$.

Since $\delta(t)=0$ for $t < 0$ it is obvious that $x(t-s)=0$ for $s > t$. In other words, when a passive system has been without driving for a long time (from $t = -\infty$ to 0), then all elements will have relaxed into a stationary state (in electric circuits all elements are discharged and no currents are flowing). In the integral we can therefore replace the upper limit by t. If we assume that $v(t) = 0$ for $t < 0$ the lower limit of integration can be replaced by 0 and we then arrive at the *convolution integral* for the calculation of $y(t)$

$$y(t) = \int_0^t v(s)x(t-s)ds. \tag{7}$$

It is now obvious that the solution $x(t)$, which is the response to the input source $v(t)=\delta(t)$ is very special. It is called the *impulse response* of the system and is denoted by $i_p(t)$.

The Laplace transform of the convolution integral $\int_0^t v(s)i_p(t-s)ds$ is simply $V(s)I_p(s)$, where $V(s)$ is the Laplace transform of $v(t)$ and $I_p(s)$ is the Laplace transform of $i_p(t)$. This simple form of the Laplace transform of the convolution integral motivates the widespread use of the Laplace transform for the calculation of the response of electric networks - and more general for the solution of linear inhomogeneous initial value problems - for different inputs. Replacing $x(t)$ by $i_p(t)$ and taking the Laplace transform of Eq. (7) we obtain the simple expression

$$Y(s) = I_p(s)V(s). \tag{8}$$

Thus, once the Laplace transform of the impulse response $I_p(s)$ is found, you can find the Laplace transform $Y(s)$ of the response to any applied driving by a simple multiplication.

2. Laplace Transformation and Impulse Response

For electrical circuits the use of Laplace Transformation and the impulse response formalism is of particular importance. As an example we consider the electric circuit in Fig. 1 in which the applied voltage $v(t)$ is zero before the time t=0, but can take any form for $t \geq 0$. We wish to calculate the responses $i_2(t)$ through the load $2R$ for several different sources $v(t)$.

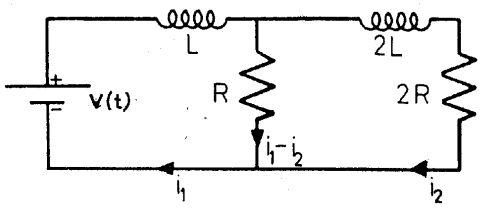

Figure 1: *Electric double loop LR circuit influenced by the voltage source* $v(t)$

2.1) Use Kirchhoff's laws to write down the dynamical equation describing the current $i_2(t)$ in the load $2R$ for $t>0$. Note that "writing down the equations" also means finding all initial conditions necessary for solving the initial value problem.

2.2) Find the Laplace transform $I_p(s)$ of the impulse response.

2.3) Use $I_p(s)$, to find $i_2(t)$ for a step-function input source, for which $v(t)=0$ for $t<0$ and $v(t)=u$ for $t \geq 0$. Plot $i_2(t)$ in a normalized diagram $0 \leq i_2 R/u \leq 0.5$ for $0 \leq \omega t \leq 30$, where $\omega = R/L$. Explain what the different parts of the solution correspond to? The important property of the circuit is its charge time, i.e., the time it takes the current i_2 to reach say 90% of its stationary maximum value. Discuss how the charge time depends on the system parameters u, R, and L.

3. Fourier Transformation

Laplace transformation is mostly used for initial value problems. We will now use another integral transformation, the Fourier transform, which is mostly used for boundary value problems. Please use the definition of the Fourier Transform used in our course book by Arfken and Weber.

3.1) Laser beams and the ground state of the linear harmonic oscillator are examples of wave functions with a Gaussian profile

$$u_0(x) = \frac{n}{\sqrt{\pi}} \exp\left(-n^2 x^2\right). \tag{9}$$

Find the Fourier transform $\tilde{u}_0(k)$ of $u_0(x)$ and discuss its symmetry properties due to the fact that the wave function $u(x)$ is real and symmetric. Here and in the following a "tilde" denotes the Fourier transformation.

3.2) Find the spread Δx of the density of the wave function in the spatial domain, $|u_0(x)|^2$, and the spread Δk of the density in the Fourier domain, $|\tilde{u}_0(k)|^2$, both defined as the Full Width at Half Maximum (FWHM). Calculate the product $\Delta x \Delta k$ and discuss the result.

3.3) Solve the 1-dimensional Schrödinger equation for a free electron

$$i\frac{\partial u}{\partial t} + \beta \frac{\partial^2 u}{\partial x^2} = 0, \quad u = u(x,t), \tag{10}$$

with the initial condition given by the Gaussian (9), using Fourier transformation.

3.4) Calculate the time-dependent maximum probability density $|u(0,t)|^2$ and FWHM Δx of the electron density $|u(x,t)|^2$. Discuss how they depend on time and the diffraction parameter β, which can be both positive and negative.

3.5) Calculate the total probability P and its counterpart in the Fourier domain $\tilde{P}$,

$$P = \int_{-\infty}^{\infty} |u(x,t)|^2 dx, \qquad P = \int_{-\infty}^{\infty} |\tilde{u}(k,t)|^2 dk, \tag{11}$$

for the solution $u(x,t)$ found in question 3.3. Discuss the relation between P and $\tilde{P}$. Show that P is conserved, i.e., that dP/dt=0, for *any wave function* $u(x,t)$ that satisfies Eq. (10) [hint: take the time derivative of P, insert $\partial u/\partial t$ from Eq. (10), and perform a partial integration].

We now consider the nonlocal Gross-Pitaevski equation

$$i\frac{\partial u}{\partial t} + \nabla^2 u - V(\vec{r}) + N(I)u = 0, \quad u = u(x,t), \tag{12}$$

which describes the collective wave function $u(x,t)$ of ultra-cold atomic gasses, or so-called Bose-Einstein Condensates (BECs). Here the parabolic potential V=$V(\vec{r}) \geq 0$ represents the magnetic trap, I=$|u|^2$ is the density of atoms, $\vec{r}$=(x,y,z), and ∇^2=$\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the 3-dimensional Laplacian. The nonlinear term N=$N(I)$ has the general nonlocal form

$$N(I) = \int R(\vec{\xi} - \vec{r})I(\vec{\xi},t)d\vec{\xi}, \qquad \int d\vec{r} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdydz \tag{13}$$

where the response function $R(\vec{r})$ accounts for the finite-range many-body interaction between the atoms. The response function $R(\vec{r})$ is assumed to be real, symmetric, and normalized, i.e., $\int R(\vec{r})d\vec{r}$=1. The nonlocal Gross-Pitaevski equation (12) conserves the total number of atoms P (because R is real) and the Hamiltonian H (because R is symmetric),

$$P = \int I d\vec{r}, \qquad H = \int \left[|\nabla u|^2 + VI - \frac{1}{2}NI\right] d\vec{r}. \tag{14}$$

In the limit when the response function becomes a delta function, $R(\vec{r})$=$\delta(\vec{r})$, the nonlinearity becomes local, $N(I)$=I. This represents the standard model for BECs, from which it is well-known that the atom cloud collapses at its center if the number of atoms exceeds a certain critical value. It will now be your task to show that a detrimental collapse cannot occur for wave functions described by Eq. (12), i.e., when nonlocality is taken into account in the general form (13). You should do this by bounding the Hamiltonian from below for a fixed number of atoms P. Through the so-called Lyapunov theory this assures that collapse cannot occur, and that the ground state is stable (if it exists).

3.6) Find $\tilde{N}$ using the convolution theorem.

3.7) Show that $\int NId\vec{r}$ is a constant times $\int \tilde{R}(\vec{k})|\tilde{I}(\vec{k})|^2 d\vec{k}$ using Parceval's relation.

3.8) Show that $|\tilde{I}(\vec{k})|^2 \leq CP^2$ using the inverse Fourier transform and Schwarz inequality [equation (9.78) in Arfken and Weber]. Find the constant C.

3.9) Show that H is bounded from below by a constant times the conserved quantity $R(\vec{0})P^2$.

For plotting you may use any software

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