Course 02647, Fall 2003
Applied Mathematics for Physicists

HOME WORK 1

Linearity, Nonlinearity, and Laplace Transformation

1. Mathematical model for a single suspension

For an engineer working with mechanical multibody systems it is important to know in depth the main basic parts of the corresponding multibody model. Here we consider the simplest example of two parts of a machine that are connected by a spring and a damper in parallel. The restoring force of the spring depends only on its deformation and the damping depends only on the relative velocity of the bodies. We further assume that the two parts are rigid bodies and that the inertia of one of the bodies is very much bigger than the inertia of the other body.

1.1) Make a simple diagram of the situation in the example, choose appropriate notations and show them on the diagram, and introduce a suitable coordinate system for the formulation of the dynamical system, that describes the motion of one of the rigid bodies.

1.2) Write down the general dynamical equations for the system using Newton's second law.

2. The linear operator

We consider the differential form

$$Lx = \frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx \tag{1}$$

where $x = x(t)$, $t > 0$, $a \in \Re$, $b \in \Re$.

2.1) Show that Lx is a linear form.

2.2) A linear differential operator is defined as a linear differential form defined on a domain Ω, which is determined by restrictions on x, $x \in \Omega$. Consider the differential equation

$$Lx = f(t)\,, \quad t > 0\,, \quad f \in \Re \tag{2}$$

where L is defined by (1), and formulate a linear initial value problem for (2), that has a unique solution $x = x(t)$. In other words - find a Ω.

3. The nonlinear operator

We consider the differential form

$$Nx = \frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx + x^3 \tag{3}$$

where $x = x(t)$, $t > 0$, $a \in \Re$, $b \in \Re$.

3.1) Show that Nx is a nonlinear form

3.2) Does the principle of superposition hold for the initial value problem $Nx = 0$, $x \in \Omega$, $\Omega : x \in C^2$? Explain why.

4. Solving the linearized problem

Consider the linearization of the dynamical system you derived in question 1.2.

4.1) Reduce it to the standard form of the equation of motion of an unforced, damped linear oscillator

$$\frac{d^2x}{dt^2} + 2\zeta\omega\frac{dx}{dt} + \omega^2 x = 0 \tag{4}$$

where $t > 0$, $\zeta > 0$, $\omega > 0$, and Ω: $x(t) \in C^2$, with the initial condition $t = 0$: $x = x_0$, $\frac{dx}{dt} = v_0$. Identify the parameters ζ and ω.

4.2) Use Laplace transformation to solve the initial value problem (4).

4.3) Find all the qualitatively different forms of motions $x(t)$ described by Eq. (4) for the initial condition x_0=1 and v_0=0 and sketch them on the same figure x versus t with $-x_0 \le x \le x_0$.

4.4) On which parameter(s) does the form of the motions found in 4.2 depend? Do the initial values have any effect on the form of the motions?

4.5) Consider the damped, forced linear oscillator given by

$$2.56\ddot{x} + 0.32\dot{x} + x = 2.5\cos t \tag{5}$$

Solve (5) analytically (directly or using Laplace transformation) and show that $x(t)$ tends towards the same harmonic function for $t \to \infty$ for arbitrary choice of initial values.

5. Solving the nonlinear problem

5.1) Consider the damped, forced nonlinear oscillator given by

$$2.56\ddot{x} + 0.32\dot{x} + x + 0.05x^3 = 2.5\cos t \tag{6}$$

Plot the solutions for the two different initial conditions

$$(x, \dot{x}) = (4.5, 13.0) \quad \text{for } t = 0 \quad \text{and}$$
$$(x, \dot{x}) = (-1.55, 0.7) \quad \text{for } t = 0$$

in the same diagram for $30 < t < 100$, using MATLAB. Why is the Laplace transform method not suitable for the solution of eqn. (6) with the appropriate initial conditions?

5.2) It is clearly seen that the two different initial conditions in 6.1 lead to quite different solutions. The two periodic equilibrium states have significantly different amplitudes although the parameters in the system - the mass of the oscillator, the damping, the restoring force and the external forcing all are the same in the two cases. This is an example of coexisting equilibrium solutions. How does that compare with the results of 4.5?