M.Sc. Course No. 02647, Fall 2002
Applied Mathematics for Physicists

SOLUTIONS TO TEST 3

1. TENSOR CALCULUS AND GROUP THEORY (estimated time ~ 15 min):

1.1) Calculate

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2, \tag{1}$$

where σ_1, σ_2, and σ_3 are the Pauli spin matrices, and show that

$$[\sigma^2, \sigma_j] = 0, \tag{2}$$

for j=1,2,3.

1.2) Let Φ be a traceless Dirac matrix. Calculate $\tan\Phi$ and ArctanΦ.

SOLUTION:

1.1) From the known relation $\sigma_i^2=I$ *(unit matrix) we find that* $\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3I$. *This gives us*

$$[\sigma^2, \sigma_j] = [3I, \sigma_j] = 3(I\sigma_j - \sigma_j I).$$

Since I *is the unit matrix we know that* $I\sigma_j=\sigma_j I$ *and thus*

$$[\sigma^2, \sigma_j] = 3(I\sigma_j - \sigma_j I) = 3(\sigma_j I - \sigma_j I) = 0.$$

1.2) Since Φ *is a traceless Dirac matrix we know (see Homework 67) that*

$$\Phi = \phi_1\sigma_1 + \phi_2\sigma_2 + \phi_3\sigma_3, \qquad \phi_n \in R$$

and the expansion for $\tan\Phi$ *is given by*

$$\tan\Phi = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n\Phi^{2n-1}}{(2n)!}$$

From Homework 67 we know that $\Phi^{2n-1} = |\Phi|^{2n-2}\Phi$, *which gives*

$$\tan\Phi = \Phi\sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n|\Phi|^{2n-2}}{(2n)!} = \frac{\Phi}{|\Phi|}\sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n|\Phi|^{2n-1}}{(2n)!} = \frac{\Phi}{|\Phi|}\tan|\Phi|$$

Likewise the expansion for ArctanΦ *is given by*

$$\text{Arctan}\Phi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\Phi^{2n-1}}{(2n-1)!} = \frac{\Phi}{|\Phi|}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}|\Phi|^{2n-1}}{(2n-1)!} = \frac{\Phi}{|\Phi|}\text{Arctan}|\Phi|$$

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2. CALCULUS OF VARIATIONS (estimated time $\sim$ 15 min.):

Consider the Lagrangian density $\mathcal{L}$=$\mathcal{L}(u, u^*, u_t, u_t^*, u_{xx}, u_{xx}^*)$, given by

$$\mathcal{L} = |u_t|^2 + |u_{xx}|^2 - \left(\frac{1}{\sigma+1}\right)|u|^{2\sigma+2}, \tag{3}$$

where $\sigma > 0$ is a real parameter and u^* is the complex conjugate of the function u=$u(x,t)$.

2.1) Use Hamilton's principle to derive the dynamical equation for $u(x,t)$ (remember that u and u^* are treated as independent functions).

SOLUTION:

2.1) According to Hamilton's principle $u(x,t)$ must satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u^*} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t^*}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial u_{xx}^*}\right) = 0.$$

We first write the Lagrangian density (3) in the form

$$\mathcal{L} = u_t u_t^* + u_{xx} u_{xx}^* - \left(\frac{1}{\sigma+1}\right)(uu^*)^{\sigma+1}.$$

The Euler-Lagrange equations then become

$$-\left(\frac{1}{\sigma+1}\right)u^{\sigma+1}(\sigma+1)(u^*)^{\sigma} - \frac{\partial}{\partial t}(u_t) + \frac{\partial^2}{\partial x^2}(u_{xx}) = 0,$$

which we can reduce to

$$u_{tt} - u_{xxxx} + |u|^{2\sigma}u = 0.$$

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3. CALCULUS OF VARIATIONS (estimated time ~ 10+5+15 min.):

Consider the eigenvalue problem

$$\frac{d}{dx}\left(\sqrt{x}\,u_x\right) + \frac{\lambda}{\sqrt{x}}u = 0, \qquad x \in [1,4], \quad u(1) = u(4) = 0, \tag{4}$$

where $u{=}u(x)$ and $u_x{=}du/dx$. For this problem the exact minimum eigenvalue is $\lambda_{\text{min}} = \pi^2/4$.

3.1) Show that requiring J, given by

$$J = \int_1^4 \sqrt{x}\,u_x^2\,dx, \tag{5}$$

to have a stationary value, subject to the constraint or normalizing condition

$$\int_1^4 \frac{u^2}{\sqrt{x}}\,dx = 5, \tag{6}$$

leads to the Sturm-Liouville equation in Eq. (4) (remember you are free to choose either $+\lambda$ or $-\lambda$ as the constant Lagrangian multiplier).

3.2) Find the constant α that makes the function

$$u(x) = x^2 - 5x + \alpha \tag{7}$$

suitable as a trial eigenfunction for the Rayleigh-Ritz variational technique.

3.3) Use the Rayleigh-Ritz variational technique with the trial eigenfunction (7) to find an approximate value for the ground-state (or minimum) eigenvalue.

SOLUTION:

3.1) The normalization (6) requires that the variation is zero:

$$\delta \int_1^4 \phi(u,x)\,dx = 0, \qquad \phi(u,x) = u^2/\sqrt{x}.$$

Combining with the variation $\delta J{=}\delta J(u_x,x){=}0$ we obtain

$$\delta \int_1^4 g(u,u_x,x)\,dx = 0, \qquad g = g(u,u_x,x) \equiv \sqrt{x}u_x^2 - \lambda u^2/\sqrt{x},$$

where λ is constant Lagrange multiplier. The new composite function g must satisfy the usual Euler-Lagrange equations

$$\frac{\partial g}{\partial u} - \frac{d}{dx}\left(\frac{\partial g}{\partial u_x}\right) = -2\lambda\frac{u}{\sqrt{x}} - \frac{d}{dx}\left(2\sqrt{x}u_x\right) = 0, \quad \Rightarrow \quad \frac{d}{dx}\left(\sqrt{x}u_x\right) + \frac{\lambda}{\sqrt{x}}u = 0,$$

which we identify as the Sturm-Liouville problem (4).

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3.2) To apply the Rayleigh-Ritz variational technique on the eigenvalue problem (4) requires that the trial function must satisfy the corresponding boundary conditions $u(1)$=$u(4)$=0:

$$\left.\begin{array}{lllll} u(1) & = & 1-5+\alpha & = & \alpha-4=0 \\ u(4) & = & 16-20+\alpha & = & \alpha-4=0 \end{array}\right\} \quad \Rightarrow \quad \alpha=4.$$

3.3) The eigenvalue problem (4) is a Sturm-Liouville problem with $p(x)=\sqrt{x}$ and $w(x)=1/\sqrt{x}$, for which the boundary contribution $[pu_x u]_1^4$=0 is zero. Thus one may use either expression for the functional $F(u,u_x,x)$ to obtain the a variational approximation λ_{t} to the minimum eigenvalue,

$$\begin{aligned} F &= \frac{\int_1^4 (x^{1/2}u_x^2)dx}{\int_1^4 (x^{-1/2}u^2)dx} = \frac{\int_1^4 [x^{1/2}(2x-5)^2]dx}{\int_1^4 [x^{-1/2}(x^2-5x+4)^2]dx} \\ &= \frac{\int_1^4 (4x^{5/2}-20x^{3/2}+25x^{1/2})dx}{\int_1^4 (x^{7/2}-10x^{5/2}+23x^{3/2}-40x^{1/2}+16x^{-1/2})dx} \\ &= \frac{[\frac{8}{7}x^{7/2}-8x^{5/2}+\frac{50}{3}x^{3/2}]_1^4}{[\frac{2}{9}x^{9/2}-\frac{20}{7}x^{7/2}+\frac{46}{5}x^{5/2}-\frac{80}{3}x^{3/2}+32x^{1/2}]_1^4} = \frac{2175}{824} = 2.64 \\ &= \lambda_{\mathrm{t}} \approx \lambda_{\min} \end{aligned}$$

The relative deviation is

$$\Delta\lambda = \frac{\lambda_{\mathrm{t}}-\lambda_{\min}}{\lambda_{\min}} = 6.98\%,$$

which is reasonable. As a check we see that $\lambda_{\mathrm{t}} > \lambda_{\min}$ as we know it should be.