TEST 3:

- **Aids: All aids allowed**
- **Time allocated: 1 hour and 30 minutes**
- **Character scale: 13-scale**
- **The test is evaluated as a whole**

1. TENSOR CALCULUS AND GROUP THEORY (estimated time ~ 15 min):

1.1) Calculate

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2, \tag{1}$$

where σ_1, σ_2, and σ_3 are the Pauli spin matrices, and show that

$$[\sigma^2, \sigma_j] = 0, \tag{2}$$

for j=1,2,3.

1.2) Let Φ be a traceless Dirac matrix. Calculate $\tan\Phi$ and $\mathrm{Arctan}\Phi$.

2. CALCULUS OF VARIATIONS (estimated time ~ 15 min.):

Consider the Lagrangian density $\mathcal{L}=\mathcal{L}(u, u^*, u_t, u_t^*, u_{xx}, u_{xx}^*)$, given by

$$\mathcal{L} = |u_t|^2 + |u_{xx}|^2 - \left(\frac{1}{\sigma+1}\right)|u|^{2\sigma+2}, \tag{3}$$

where $\sigma > 0$ is a real parameter and u^* is the complex conjugate of the function u=$u(x,t)$.

2.1) Use Hamilton's principle to derive the dynamical equation for $u(x,t)$ (remember that u and u^* are treated as independent functions).

Turn page

3. CALCULUS OF VARIATIONS (estimated time $\sim$ 10+5+15 min.):

Consider the eigenvalue problem

$$\frac{d}{dx}\left(\sqrt{x}\, u_x\right) + \frac{\lambda}{\sqrt{x}}\, u = 0, \qquad x \in [1,4], \quad u(1) = u(4) = 0, \tag{4}$$

where $u=u(x)$ and $u_x=du/dx$. For this problem the exact minimum eigenvalue is $\lambda_{\text{min}} = \pi^2/4$.

3.1) Show that requiring J, given by

$$J = \int_1^4 \sqrt{x}\, u_x^2\, dx, \tag{5}$$

to have a stationary value, subject to the constraint or normalizing condition

$$\int_1^4 \frac{u^2}{\sqrt{x}}\, dx = 5, \tag{6}$$

leads to the Sturm-Liouville equation in Eq. (4) (remember you are free to choose either $+\lambda$ or $-\lambda$ as the constant Lagrangian multiplier).

3.2) Find the constant α that makes the function

$$u(x) = x^2 - 5x + \alpha \tag{7}$$

suitable as a trial eigenfunction for the Rayleigh-Ritz variational technique.

3.3) Use the Rayleigh-Ritz variational technique with the trial eigenfunction (7) to find an approximate value for the ground-state (or minimum) eigenvalue.