SOLUTIONS TO TEST 1

LINEARITY:

1) Consider the differential form

$$Lu = u'' + cu^{p-1}, \tag{1}$$

where $u = u(t) \in C^2$, $t > 0$, $c \in \Re$, and p is an integer. Here and in all other questions "prime" denotes differentiation with respect to the argument, i.e. $u'(t)$=du/dt.

1.1) For which values of the constants c and p is Lu a linear form? (just give the values - do not show calculations).

SOLUTION:

1.1) Lu is a linear form if $L(a_1u_1+a_2u_2) = a_1Lu_1+a_2Lu_2$ *for arbitrary values of the constants* a_1 *and* a_2*. Thus Lu is a linear form for the following values of c and p:*

$$\begin{aligned} c = 0 \quad &\Rightarrow \quad p \in Z \\ p = 2 \quad &\Rightarrow \quad c \in \Re \end{aligned}$$

Note that even though the form (1) might seem linear for $(p = 1, c \neq 0)$*, the fact that it is inhomogeneous in this case means that it does not satisfy the general criterion for linearity.*

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LAPLACE TRANSFORMATION:

2) Consider the linear differential equation

$$u'''' + 2u'' + u = f(t), \tag{2}$$

where $u{=}u(t) \in C^4$, $t > 0$, with the initial condition $u(0) = u''(0){=}0$, $u'(0){=}1$, $u'''(0){=}{-4}$.

2.1) Calculate the impulse response function $u_p(t)$ using Laplace transformation.

SOLUTION:

2.1) To use the impulse function technique requires homogeneous initial conditions. To solve Eq. (2) we therefore define two new functions

$$u(t) = u_h(t) + \phi(t), \quad u_h'''' + 2u_h'' + u_h = 0,$$

where the homogeneous solution $u_h(t)$ satisfies the same initial conditions as $u(t)$,

$$u_h(0) = u(0), \quad u_h'(0) = u'(0), \quad u_h''(0) = u''(0), \quad u_h'''(0) = u'''(0).$$

By construction $\phi(t)$ satisfies the same Eq. (2), but with homogeneous initial conditions,

$$\phi'''' + 2\phi'' + \phi = f(t), \quad \phi(0) = \phi'(0) = \phi''(0) = \phi'''(0) = 0. \tag{3}$$

The impulse response $\phi_p(t)$ is found by letting $f(t)$ be a δ-function in Eq.(3). Thus $\phi_p(t)$ is the solution to the following equation

$$\phi_p'''' + 2\phi_p'' + \phi_p = \delta(t), \quad \phi_p(0) = \phi_p'(0) = \phi_p''(0) = \phi_p'''(0) = 0, \tag{4}$$

also with homogeneous initial conditions. Laplace transformation of this equation gives

$$[s^4 + 2s^2 + 1]\mathcal{L}\{\phi_p\} = 1,$$

where we have used that all contributions from the initial conditions are zero. This gives

$$\mathcal{L}\{\phi_p\} = \frac{1}{(s^2+1)^2} = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{s^2-1}{(s^2+1)^2}\right]$$

The first simple expression can be found in Schaum's, the second in Table 15.2 in the course book. We then find the impulse response

$$\phi_p(t) = \frac{1}{2}\left[\sin(t) - t\cos(t)\right].$$

Note that $\phi_p(0) = \phi_p'(0) = \phi_p''(0){=}0$ as they should, but $\phi_p'''(0){=}1$. This is in fact natural from looking at Eq. (4). Integrating once just around t=0 (with ϕ_p'' and ϕ_p continuous in t=0) one obtains exactly $\phi_p'''(0){=}1$. This is no contradiction. To understand one should look more careful at how the limit $t \to 0$ is taken, i.e., consider that the δ-function is "turned on" at $t = 0^+$. A good way to do this would be to first consider $\delta(t - t_0)$ and then let $t_0 \to 0$ from the positive side. In other words, the impulse $\delta(t)$ at t=0 changes instantaneously ϕ_p''' from 0 to 1 (see also example 15.9.3 in the course book).

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FOURIER TRANSFORMATION:

3) Consider the linear differential equation

$$u - \sigma^2 u'' = f(x), \tag{5}$$

where $\sigma{>}0$ is a real constant and $u{=}u(x){\in}C^2$ is a localized function [$u(\pm\infty){=}u'(\pm\infty){=}0$]. The given function $f(x)$ is also localized, i.e., both $u(x)$ and $f(x)$ have a Fourier transform. In the following you must use the definition of the Fourier Transform $\tilde{u}(k)$ and its inverse

$$\tilde{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \mathrm{e}^{ikx} dx, \quad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k) \mathrm{e}^{-ikx} dk. \tag{6}$$

3.1) The solution to Eq. (5) can be written in the form

$$u(x) = \int_{-\infty}^{\infty} R(x - x_1) f(x_1) dx_1, \tag{7}$$

where the response function $R(x)$ is real. Use Fourier transformation and the convolution theorem to solve Eq. (5) and show that $R(x) = \frac{1}{2\sigma}\exp(-|x|/\sigma)$.
Hint: $\int_0^\infty \frac{\cos(mx)}{x^2+a^2} dx = \frac{\pi}{2a}\exp(-|ma|)$.

SOLUTION:

3.1) Fourier transformation of Eq. (5) gives

$$(1 + \sigma^2 k^2)\tilde{u}(k) = \tilde{f}(k), \quad \Rightarrow \quad \tilde{u}(k) = \left(\frac{1}{1 + \sigma^2 k^2}\right) \tilde{f}(k) \equiv \tilde{Q}(k)\tilde{f}(k),$$

where we have defined a new function $Q(x)$, given by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp(-ikx)}{1 + \sigma^2 k^2} dk = \frac{1}{\sigma^2}\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos(xk)}{k^2 + \sigma^{-2}} dk = \frac{\sqrt{2\pi}}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right).$$

With the definition (6) of the Fourier transform pair the convolution theorem states that

$$\tilde{u}(k) = \tilde{Q}(k)\tilde{f}(k) \quad \Rightarrow \quad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(x - x_1) f(x_1) dx_1.$$

Comparing with the form (7) we identify $R(x) = Q(x)/\sqrt{2\pi}$, which gives the following correct expression for $R(x)$

$$R(x) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right).$$

SOLUTIONS TO TEST 1

STURM-LIOUVILLE THEORY:

4) Consider the eigenvalue problem

$$(\sqrt{x}\,u')' + \frac{\lambda}{\sqrt{x}}\,u = 0, \quad u(1) = u(4) = 0, \tag{8}$$

where $u = u(x) \in C^2$ and $x \in [1, 4]$.

4.1) Is λ=0 an eigenvalue?

4.2) Equation (8) has the complete solution $u(x) = A\sin(2\sqrt{\lambda x}) + B\cos(2\sqrt{\lambda x})$ for $\lambda \neq 0$. Find all eigenvalues and eigenfunctions.

SOLUTION:

4.1) The eigenvalue problem (8) is a Sturm-Liouville problem, which we write in standard form

$$\hat{L}u \equiv -(x^{1/2}u')' = \lambda x^{-1/2}u, \quad u(1) = u(4) = 0.$$

For λ=0 Eq. (8) reduces to

$$(x^{1/2}u')' = 0 \quad \Rightarrow \quad u' = c_0 x^{-1/2} \quad \Rightarrow \quad u(x) = 2c_0 x^{1/2} + c_1,$$

where $c_{0,1}$ are constants of integration. Checking the boundary conditions we find that

$$u(1) = 2c_0 + c_1 = 0 \quad \text{and} \quad u(4) = 4c_0 + c_1 = 0 \quad \Rightarrow \quad c_0 = c_1 = 0.$$

Since the only solution for λ=0 is the trivial zero-solution, λ=0 is <u>not</u> an eigenvalue.

4.2) For $\lambda \neq 0$ the boundary conditions for the given solution become

$$u(1) = A\sin(2\sqrt{\lambda}) + B\cos(2\sqrt{\lambda}) = 0,$$
$$u(4) = A\sin(4\sqrt{\lambda}) + B\cos(4\sqrt{\lambda}) = 0.$$

We can write this in matrix form

$$\begin{bmatrix} \sin(2\sqrt{\lambda}) & \cos(2\sqrt{\lambda}) \\ \sin(4\sqrt{\lambda}) & \cos(4\sqrt{\lambda}) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For nontrivial solutions to exist the determinant must be zero, which requires

$$\sin(2\sqrt{\lambda})\cos(4\sqrt{\lambda}) - \cos(2\sqrt{\lambda})\sin(4\sqrt{\lambda}) = \sin(2\sqrt{\lambda} - 4\sqrt{\lambda}) = -\sin(2\sqrt{\lambda}) = 0.$$

For $\lambda < 0$ the requirement is $\sinh(2\sqrt{\lambda})$=0, which cannot be fulfilled. Thus $\lambda < 0$ is <u>not</u> an eigenvalue. For $\lambda > 0$ the requirement is $\sin(2\sqrt{\lambda})$=0, which gives the eigenvalues

$$2\sqrt{\lambda} = n\pi \quad \Rightarrow \quad \lambda_n = (n\pi/2)^2, \quad n = 1, 2, 3, \ldots.$$

The eigenfunctions then become $u_n(x) = A\sin(n\pi x) + B\cos(n\pi x)$. From the boundary conditions we find B=0, and thus the eigenfunctions are given by

$$u_n(x) = A\sin(n\pi\sqrt{x}),$$

where the constant A is arbitrary.