TEST 1:

- **Aids: All aids allowed**
- **Time allocated: 1 hour**
- **Character scale: 13-scale**
- **The test is evaluated as a whole**

LINEARITY:

1) Consider the differential form

$$Lu = u'' + cu^{p-1}, \tag{1}$$

where $u = u(t) \in C^2$, $t > 0$, $c \in \Re$, and p is an integer. Here and in all other questions "prime" denotes differentiation with respect to the argument, i.e. $u'(t)=du/dt$.

1.1) For which values of the constants c and p is Lu a linear form? (just give the values - do not show calculations).

LAPLACE TRANSFORMATION:

2) Consider the linear differential equation

$$u'''' + 2u'' + u = f(t), \tag{2}$$

where $u=u(t) \in C^4$, $t > 0$, with the initial condition $u(0) = u''(0)=0$, $u'(0)=1$, $u'''(0)=-4$.

2.1) Calculate the impulse response function $u_p(t)$ using Laplace transformation.

FOURIER TRANSFORMATION:

3) Consider the linear differential equation

$$u - \sigma^2 u'' = f(x), \tag{3}$$

where $\sigma{>}0$ is a real constant and $u{=}u(x){\in}C^2$ is a localized function $[u(\pm\infty){=}u'(\pm\infty){=}0]$. The given function $f(x)$ is also localized, i.e., both $u(x)$ and $f(x)$ have a Fourier transform. In the following you must use the definition of the Fourier Transform $\tilde{u}(k)$ and its inverse

$$\tilde{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) \mathrm{e}^{ikx} dx, \quad u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k) \mathrm{e}^{-ikx} dk. \tag{4}$$

3.1) The solution to Eq. (3) can be written in the form

$$u(x) = \int_{-\infty}^{\infty} R(x - x_1) f(x_1) dx_1, \tag{5}$$

where the response function $R(x)$ is real. Use Fourier transformation and the convolution theorem to solve Eq. (3) and show that $R(x) = \frac{1}{2\sigma} \exp(-|x|/\sigma)$.
Hint: $\int_0^\infty \frac{\cos(mx)}{x^2+a^2} dx = \frac{\pi}{2a} \exp(-|ma|)$.

STURM-LIOUVILLE THEORY:

4) Consider the eigenvalue problem

$$(\sqrt{x}\, u')' + \frac{\lambda}{\sqrt{x}} u = 0, \quad u(1) = u(4) = 0, \tag{6}$$

where $u = u(x) \in C^2$ and $x \in [1, 4]$.

4.1) Is $\lambda{=}0$ an eigenvalue?

4.2) Equation (6) has the complete solution $u(x) = A \sin(2\sqrt{\lambda x}) + B \cos(2\sqrt{\lambda x})$ for $\lambda \neq 0$. Find all eigenvalues and eigenfunctions.