Chapter 8 - Solutions to exercises

Exercises: 3, 5, 9

Exercise 3

We have that

$$\nu_t(i,j;\mathbf{x}^{(t)}) = \Pr(\mathbf{X}^{(t)} = \mathbf{x}^{(t)}, C_{t-1} = i, C_t = j),$$

for $i,j \in \{1,\dots,m\}$ and $t \geq 2$, which are the forward probabilities for a second-order HMM. We also use the following short-hand notation

$$\begin{aligned} u(i,j) &= \Pr(C_{t-1} = i, C_t = j) \\ p_i(x_t) &= \Pr(X_t = x_t | C_t = i) \\ \gamma(i,j,k) &= \Pr(C_t = k | C_{t-1} = j, C_{t-2} = i). \end{aligned}$$

Then

$$\nu_2(i,j;\mathbf{x}^{(t)}) = u(i,j) p_i(x_1) p_j(x_2).$$

a)

Using the above we get for $t \geq 3$:

$$\begin{aligned} &\left(\sum_{i=1}^{m} \nu_{t-1}(i,j;\mathbf{x}^{(t-1)})\gamma(i,j,k)\right) p_k(x_t) \\ =&\left(\sum_{i=1}^{m} \Pr(\mathbf{X}^{(t-1)} = \mathbf{x}^{(t-1)}, C_{t-2} = i, C_{t-1} = j)\Pr(C_t = k | C_{t-1} = j, C_{t-2} = i)\right) \Pr(X_t = x_t | C_t = k) \\ =&\left(\sum_{i=1}^{m} \Pr(\mathbf{X}^{(t-1)} = \mathbf{x}^{(t-1)}, C_{t-2} = i, C_{t-1} = j, C_t = k)\right) \Pr(X_t = x_t | C_t = k) \\ =&\Pr(\mathbf{X}^{(t-1)} = \mathbf{x}^{(t-1)}, C_{t-1} = j, C_t = k)\Pr(X_t = x_t | C_t = k) \\ =&\Pr(\mathbf{X}^{(t)} = \mathbf{x}^{(t)}, C_{t-1} = j, C_t = k) \\ =&\nu_t(j,k;\mathbf{x}^{(t)}) \end{aligned}$$

b)

We have that

$$\nu_T(i,j;\mathbf{x}^{(T)}) = \Pr(\mathbf{X}^{(T)} = \mathbf{x}^{(T)}, C_{T-1} = i, C_T = j),$$

thus

$$L_T = \Pr(\mathbf{X}^{(T)} = \mathbf{x}^{(T)}) = \sum_{i=1}^{m}\sum_{j=1}^{m} \nu_T(i, j; \mathbf{x}^{(T)}).$$

c)

Calculating the likelihood involves calculating $\nu_T(i, j; \mathbf{x}^{(T)})$ for all combinations of i and j, i.e. m^2 times. Calculating $\nu_T(i, j; \mathbf{x}^{(T)})$ requires a recursion with T steps where each step is a sum of m terms. So, the total computational effort for calculating the likelihood is proportional to $m^2Tm = Tm^3$.

Exercise 5

Assuming stationarity of the underlying Markov chain the following ingredients are required to calculate the auto-correlation function

$$\begin{aligned}
\mathrm{E}(X_t) &= \sum_{i=m}^{m} \delta_i \mathrm{E}(X_t|C_t = i) \\
\mathrm{E}(X_t^2) &= \sum_{i=m}^{m} \delta_i \mathrm{E}(X_t^2|C_t = i) = \sum_{i=m}^{m} \delta_i (\mathrm{Var}(X_t|C_t = i) + \mathrm{E}(X_t|C_t = i)^2) \\
\mathrm{Var}(X_t) &= \mathrm{E}(X_t^2) - (\mathrm{E}(X_t))^2 \\
\mathrm{E}(X_t X_{t+k}) &= \sum_{i=m}^{m} \sum_{j=m}^{m} \delta_i \gamma_{ij}(k) \mathrm{E}(X_t|C_t = i) \mathrm{E}(X_{t+k}|C_{t+k} = j)
\end{aligned}$$

a)

We consider the case where the state dependent distributions are normal, i.e.

$$X_t|C_t = i \sim N(\mu_i, \sigma_i^2).$$

The auto-covariance function is

$$\begin{aligned}
\mathrm{Cov}(X_t, X_{t+k}) &= \mathrm{E}(X_t X_{t+k}) - \mathrm{E}(X_t)\mathrm{E}(X_{t+k}) \\
&= \left(\sum_{i=m}^{m} \sum_{j=m}^{m} \delta_i \gamma_{ij}(k) \mu_i \mu_j \right) - \left(\sum_{i=1}^{m} \delta_i \mu_i \right)^2 \\
&= \boldsymbol{\delta} \boldsymbol{\Lambda} \boldsymbol{\Gamma}^k \boldsymbol{\mu}' - (\boldsymbol{\delta} \boldsymbol{\mu}')^2,
\end{aligned}$$

where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, $\boldsymbol{\Lambda} = \mathrm{diag}(\mu_1, \ldots, \mu_m)$, and $\boldsymbol{\Gamma}^k$ is the k-step transition probability matrix.

The auto-correlation function is then

$$\begin{aligned}
\mathrm{Corr}(X_t, X_{t+k}) &= \frac{\mathrm{Cov}(X_t, X_{t+k})}{\mathrm{Var}(X_t)} \\
&= \frac{\boldsymbol{\delta} \boldsymbol{\Lambda} \boldsymbol{\Gamma}^k \boldsymbol{\mu}' - (\boldsymbol{\delta} \boldsymbol{\mu}')^2}{\boldsymbol{\delta} \boldsymbol{\Lambda} \boldsymbol{\mu}' + \boldsymbol{\delta} \boldsymbol{\sigma}' - (\boldsymbol{\delta} \boldsymbol{\mu}')^2},
\end{aligned}$$

where $\boldsymbol{\sigma} = (\sigma_1^2, \ldots, \sigma_m^2)$.

b)

Now consider the case where the state dependent distributions are binomial, i.e.

$$X_t|C_t = i \sim \text{Bin}(n, p_i).$$

Note that the number of trials n is assumed to be known and constant over i and t.

The auto-covariance function is

$$\begin{aligned}\text{Cov}(X_t, X_{t+k}) &= \text{E}(X_t X_{t+k}) - \text{E}(X_t)\text{E}(X_{t+k}) \\ &= \left(\sum_{i=m}^{m}\sum_{j=m}^{m} \delta_i \gamma_{ij}(k) n p_i n p_j\right) - \left(\sum_{i=1}^{m} \delta_i n p_i\right)^2 \\ &= n^2[\boldsymbol{\delta}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^k \boldsymbol{p}' - (\boldsymbol{\delta}\boldsymbol{p}')^2],\end{aligned}$$

where $\boldsymbol{p} = (p_1, \ldots, p_m)$, $\boldsymbol{\Lambda} = \text{diag}(p_1, \ldots, p_m)$. So, the auto-correlation function is

$$\begin{aligned}\text{Corr}(X_t, X_{t+k}) &= \frac{\text{Cov}(X_t, X_{t+k})}{\text{Var}(X_t)} \\ &= \frac{n^2[\boldsymbol{\delta}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^k \boldsymbol{p}' - (\boldsymbol{\delta}\boldsymbol{p}')^2]}{n[\boldsymbol{\delta}\boldsymbol{\Lambda}\boldsymbol{p}'(n-1) + \boldsymbol{\delta}\boldsymbol{p}' - n(\boldsymbol{\delta}\boldsymbol{p}')^2]} \\ &= \frac{n[\boldsymbol{\delta}\boldsymbol{\Lambda}\boldsymbol{\Gamma}^k \boldsymbol{p}' - (\boldsymbol{\delta}\boldsymbol{p}')^2]}{\boldsymbol{\delta}\boldsymbol{\Lambda}\boldsymbol{p}'(n-1) + \boldsymbol{\delta}\boldsymbol{p}' - n(\boldsymbol{\delta}\boldsymbol{p}')^2}.\end{aligned}$$

Exercise 9

A Binomial-HMM $\{X_t\}$ has transition probability matrix

$$\mathbf{\Gamma} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

The state dependent distributions are Binomial with parameters 2 and 0/0.5/1 in the three states respectively.

To show that $\{X_t\}$ is an irreversible process we calculate $\Pr(X_t = 0, X_{t+1} = 1)$ and $\Pr(X_t = 0, X_{t+1} = 1)$. First recall from page 18 in Zucchini09 that $\boldsymbol{\delta} = \frac{1}{32}(15, 9, 8)$. Then

$$\Pr(X_t = 0, X_{t+1} = 1) = \boldsymbol{\delta}\mathbf{P}(0)\mathbf{\Gamma}\mathbf{P}(1)\mathbf{1}' = 0.078125,$$
$$\Pr(X_t = 1, X_{t+1} = 0) = \boldsymbol{\delta}\mathbf{P}(1)\mathbf{\Gamma}\mathbf{P}(0)\mathbf{1}' = 0.09375.$$

Since the two probabilities are not equal the HMM is irreversible.

R-code:

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G = rbind(c(1/3,1/3,1/3),c(2/3,0,1/3),c(1/2,1/2,0))
del = c(15,9,8)/32

n = 2

P0 = diag(c(dbinom(0,n,0),dbinom(0,n,0.5),dbinom(0,n,1)))
P1 = diag(c(dbinom(1,n,0),dbinom(1,n,0.5),dbinom(1,n,1)))

sum(del %*% P0 %*% G %*% P1)

sum(del %*% P1 %*% G %*% P0)
```