Chapter 2 - Solutions to exercises

Exercises: 1,2,6,9,11

Exercise 1

Stationary two-state Poisson with

$$\mathbf{\Gamma} = \begin{pmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\lambda} = (1, 3)$$

Compute $\Pr(X_1 = 0, X_2 = 2, X_3 = 1)$ in two ways.

In both cases the stationary distribution is required. Using the formula on p. 19 in the book and solving for $\boldsymbol{\delta}$ we get:

$$\boldsymbol{\delta} = \begin{pmatrix} \frac{4}{13} & \frac{9}{13} \end{pmatrix}.$$

a)

We consider all the eight (2^3) possible state sequences that can occur in a two-state Markov chain in three steps. We compute the probability of each sequence and then the probability of observing $(X_1 = 0, X_2 = 2, X_3 = 1)$ conditional on each sequence. The sequences are shown in the table below.

i	j	k	$p_i(0)$	$p_j(2)$	$p_k(1)$	δ_i	γ_{ij}	γ_{jk}	product
1	1	1	0.3679	0.1839	0.3679	$\frac{4}{13}$	$\frac{1}{10}$	$\frac{1}{10}$	$7.66 \cdot 10^{-5}$
1	1	2	0.3679	0.1839	0.1494	$\frac{4}{13}$	$\frac{1}{10}$	$\frac{9}{10}$	$2.80 \cdot 10^{-4}$
1	2	1	0.3679	0.2240	0.3679	$\frac{4}{13}$	$\frac{9}{10}$	$\frac{4}{10}$	$3.36 \cdot 10^{-3}$
1	2	2	0.3679	0.2240	0.1494	$\frac{4}{13}$	$\frac{9}{10}$	$\frac{6}{10}$	$2.05 \cdot 10^{-3}$
2	1	1	0.0498	0.1839	0.3679	$\frac{9}{13}$	$\frac{4}{10}$	$\frac{1}{10}$	$9.33 \cdot 10^{-5}$
2	1	2	0.0498	0.1839	0.1494	$\frac{9}{13}$	$\frac{4}{10}$	$\frac{9}{10}$	$3.41 \cdot 10^{-4}$
2	2	1	0.0498	0.2240	0.3679	$\frac{9}{13}$	$\frac{6}{10}$	$\frac{4}{10}$	$6.82 \cdot 10^{-4}$
2	2	2	0.0498	0.2240	0.1494	$\frac{9}{13}$	$\frac{6}{10}$	$\frac{6}{10}$	$4.15 \cdot 10^{-4}$
Total probability									0.0073

b)

Now we apply the formula $\Pr(X_1 = 0, X_2 = 2, X_3 = 1) = \boldsymbol{\delta}\mathbf{P}(0)\mathbf{\Gamma}\mathbf{P}(2)\mathbf{\Gamma}\mathbf{P}(1)\mathbf{1}'$. The $\mathbf{P}(s)$ matrices are:

$$\mathbf{P}(0) = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-3} \end{pmatrix}$$

$$\mathbf{P}(1) = \begin{pmatrix} e^{-1} & 0 \\ 0 & 3e^{-3} \end{pmatrix}$$

$$\mathbf{P}(2) = \begin{pmatrix} \frac{e^{-1}}{2} & 0 \\ 0 & \frac{3^2 e^{-3}}{2} \end{pmatrix}$$

Carrying out the calculations one obtains:

$$\Pr(X_1 = 0, X_2 = 2, X_3 = 1) = \boldsymbol{\delta}\mathbf{P}(0)\boldsymbol{\Gamma}\mathbf{P}(2)\boldsymbol{\Gamma}\mathbf{P}(1)\mathbf{1}' = 0.0073$$

This result is consistent with the result in a).

Exercise 2

With the model in exercise 1 we now observe $(X_1 = 0, X_3 = 1)$. This is a missing data scenario since we did not receive an observation related to time $t = 2$.

a)

Considering all possible sequences:

i	j	k	$p_i(0)$	$p_j(2)$	$p_k(1)$	δ_i	γ_{ij}	γ_{jk}	product
1	1	1	0.3679	0.1839	0.3679	$\frac{4}{13}$	$\frac{1}{10}$	$\frac{1}{10}$	$4.16 \cdot 10^{-4}$
1	1	2	0.3679	0.1839	0.1494	$\frac{4}{13}$	$\frac{1}{10}$	$\frac{9}{10}$	$1.52 \cdot 10^{-3}$
1	2	1	0.3679	0.2240	0.3679	$\frac{4}{13}$	$\frac{9}{10}$	$\frac{4}{10}$	$1.50 \cdot 10^{-2}$
1	2	2	0.3679	0.2240	0.1494	$\frac{4}{13}$	$\frac{9}{10}$	$\frac{6}{10}$	$9.13 \cdot 10^{-3}$
2	1	1	0.0498	0.1839	0.3679	$\frac{9}{13}$	$\frac{4}{10}$	$\frac{1}{10}$	$5.07 \cdot 10^{-4}$
2	1	2	0.0498	0.1839	0.1494	$\frac{9}{13}$	$\frac{4}{10}$	$\frac{9}{10}$	$1.85 \cdot 10^{-3}$
2	2	1	0.0498	0.2240	0.3679	$\frac{9}{13}$	$\frac{6}{10}$	$\frac{4}{10}$	$3.04 \cdot 10^{-3}$
2	2	2	0.0498	0.2240	0.1494	$\frac{9}{13}$	$\frac{6}{10}$	$\frac{6}{10}$	$1.85 \cdot 10^{-3}$
Total probability									0.0333

b)

Now we apply the formula $\Pr(X_1 = 0, X_3 = 1) = \boldsymbol{\delta}\mathbf{P}(0)\boldsymbol{\Gamma}^2\mathbf{P}(1)\mathbf{1}'$. The $\mathbf{P}(s)$ matrices are the same as in exercise 1. Carrying out the calculations one obtains:

$$\Pr(X_1 = 0, X_3 = 1) = \boldsymbol{\delta}\mathbf{P}(0)\boldsymbol{\Gamma}^2\mathbf{P}(1)\mathbf{1}' = 0.0333$$

This result is consistent with the result in a).

Exercise 6

We have the general expression for the likelihood of an HMM:

$$L_T = \boldsymbol{\delta}\mathbf{P}(x_1)\boldsymbol{\Gamma}\mathbf{P}(x_2)\cdots\boldsymbol{\Gamma}\mathbf{P}(x_T)\mathbf{1}' \tag{1}$$

In the case where the underlying hidden Markov chain degenerates to a sequence of independent random variables we have that $\Pr(C_{t+1} = j|C_t = i) = \Pr(C_{t+1} = j)$. If $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_m)$, the transition probability matrix is therefore

$$\boldsymbol{\Gamma} = \begin{pmatrix} \delta_1 & \delta_2 & \cdots & \delta_m \\ \delta_1 & \delta_2 & \cdots & \delta_m \\ \vdots & \vdots & \ddots & \delta_m \\ \delta_1 & \delta_2 & \cdots & \delta_m \end{pmatrix}.$$

Using this expression for $\boldsymbol{\Gamma}$ (1) can be shown to equal Equation (1.1) by expanding the matrix-vector multiplications.

Exercise 9

```
# Chapter 2, R-code for exercise 9, mwp 25/1-2011

statdist <- function(gamma){
  m = dim(gamma)[1]
  matrix(1,1,m) %*% solve(diag(1,m) - gamma + matrix(1,m,m))
}


poisHMM.moments <- function(m,lambda,gamma,lag.max=10){
  del = statdist(gamma)
  muHMM  = sum(del*lambda)
  varHMM = sum((del %*% diag(lambda))*lambda) + muHMM - muHMM^2
  acfHMM = matrix(0,1,lag.max+1)
  acfHMM[1] = 1
  gammapow = diag(m)
  for(i in 2:lag.max){
    gammapow = gammapow %*% gamma
    acfHMM[i] = ( sum((del %*% diag(lambda) %*% gammapow) *lambda) - muHMM^2) / varHMM
  }
  list(muHMM=muHMM,varHMM=varHMM,acfHMM=acfHMM)
}

# Setup a test HMM
gamma = rbind(c(0.8,0.2,0),c(0.05,0.7,0.25),c(0.1,0.25,0.65))
lambda = c(1,5,10)
m = length(lambda)
lag.max = 15

mom <- poisHMM.moments(m,lambda,gamma,lag.max)
```

Exercise 11

```
# Chapter 2, R-code for exercise 11, mwp 25/1-2011

statdist <- function(gamma){
  m = dim(gamma)[1]
  matrix(1,1,m) %*% solve(diag(1,m) - gamma + matrix(1,m,m))
}

poisHMM.moments <- function(m,lambda,gamma,lag.max=10){
  del = statdist(gamma)
  muHMM  = sum(del*lambda)
  varHMM = sum((del %*% diag(lambda))*lambda) + muHMM - muHMM^2
  acfHMM = matrix(0,1,lag.max+1)
  acfHMM[1] = 1
  gammapow = diag(m)
  for(i in 2:lag.max){
    gammapow = gammapow %*% gamma
    acfHMM[i] = ( sum((del %*% diag(lambda) %*% gammapow) *lambda) - muHMM^2) / varHMM
  }
  list(muHMM=muHMM,varHMM=varHMM,acfHMM=acfHMM)
}

pois.HMM.generate_sample <- function(n,m,lambda,gamma,delta=NULL){
 if(is.null(delta))delta<-solve(t(diag(m)-gamma+1),rep(1,m))
 mvect <- 1:m
 state <- numeric(n)
 state[1] <- sample(mvect,1,prob=delta)
 for (i in 2:n)
   state[i]<-sample(mvect,1,prob=gamma[state[i-1],])
 x <- rpois(n,lambda=lambda[state])
 x
}

# Setup a test HMM
gamma = rbind(c(0.8,0.2,0),c(0.05,0.7,0.25),c(0.1,0.25,0.65))
lambda = c(1,5,10)
m = length(lambda)
lag.max = 15

n = 10000

x = pois.HMM.generate_sample(n,m,lambda,gamma)

mome = list()
mome$mu = mean(x)
mome$var = var(x)
mome$acf = acf(x)

mom <- poisHMM.moments(m,lambda,gamma,lag.max)
```