# Chapter 1 - Solutions to exercises

Exercises: 3,6,8,10,(12,15).

## Exercise 3

```
# Chapter 1, R-code for exercise 3, mwp 21/1-2011

# Define model values for a 3-state mixture model
m = 3
la1 = 1
la2 = 3
la3 = 7
del = matrix(0,1,m-1)
del[1] = 0.5
del[2] = 0.2
del[3] = 1 - (del[1]+del[2])
cd = cumsum(del)

# Generate random data for a m mixture model
N = 100 # number of data points
unifvec = runif(N)
d1 = rpois(sum(unifvec < cd[1]),la1)
d2 = rpois(sum(unifvec > cd[1] & unifvec < cd[2]),la2)
d3 = rpois(sum(unifvec > cd[2]),la3)
data = c(d1,d2,d3) # Data vector

# Functions for parameter transformation
logit <- function(vec) log(vec/(1-sum(vec)))
invlogit <- function(vec) exp(vec)/(1+sum(exp(vec)))

# Make function for the negative log-likelihood
f <- function(PAR) {
  M = length(PAR)
  m = ceiling(M/2)
  LA = exp(PAR[1:m]) # transform lambdas
  DELs = invlogit(PAR[(m+1):M]) # transform deltas
  DEL = c(DELs,1-sum(DELs))
  # Equation (1.1) on p. 9
  L = DEL[1]*dpois(data,LA[1])
  for(i in 2:m){
    L = L+DEL[i]*dpois(data,LA[i])
  }
  -sum(log(L))
}

# Define starting guess for optimization
par = c(2,4,7,0.5,0.2)
PAR = par
PAR[1:3] = log(par[1:3])
PAR[4:5] = logit(par[4:5])

# Optimize using nlm
```

```
res = nlm(f,PAR)

# Back transform results
lambdas = exp(res$estimate[1:3])
deltas = invlogit(res$estimate[4:5])
```

## Exercise 6

$$\mathbf{\Gamma} = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad \boldsymbol{\delta} = (\delta_1, \delta_2)$$

### a)

Stationarity implies

$$\begin{aligned} \boldsymbol{\delta}\mathbf{\Gamma} &= \boldsymbol{\delta} \\ &\Leftrightarrow \\ \delta_1\gamma_{11} + \delta_2\gamma_{21} &= \delta_1 \\ \delta_1\gamma_{21} + \delta_2\gamma_{22} &= \delta_2 \end{aligned}$$

The stationary distribution is found by replacing one of the equations in the system with $\sum_i \delta_i = 1$ and then solving for $\boldsymbol{\delta}$. So, we have

$$\begin{aligned} \delta_1\gamma_{11} + \delta_2\gamma_{21} &= \delta_1, \\ \delta_1 + \delta_2 &= 1, \end{aligned}$$

and therefore

$$\begin{aligned} \delta_1 &= \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}}, \\ \delta_2 &= \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}}. \end{aligned}$$

### b)

Now consider

$$\mathbf{\Gamma} = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}.$$

The two sequences have the respective probabilities:

$$\begin{aligned} \Pr(\text{Seq } 1) &= \gamma_{11}\gamma_{11}\gamma_{12}\gamma_{22}\gamma_{21} = 0.01296 \\ \Pr(\text{Seq } 2) &= \gamma_{21}\gamma_{11}\gamma_{12}\gamma_{21}\gamma_{11} = 0.0396 \end{aligned}$$

The sequences have different probabilities because they consist of different transitions between the states of the Markov process. In other words: the ordering of the sequences matters, i.e. the numbers in the sequences are not independent!

# Exercise 8

## a)

We have

$$\begin{aligned}\boldsymbol{\delta}(\mathbf{B}I_m - \boldsymbol{\Gamma} + \mathbf{B}U) &= \mathbf{B}1 \\ \underbrace{\boldsymbol{\delta} - \boldsymbol{\delta}\boldsymbol{\Gamma}}_{A} + \underbrace{\boldsymbol{\delta}\mathbf{B}U}_{B} &= \mathbf{B}1\end{aligned}$$

Term $A$ is zero if $\boldsymbol{\delta}$ is invariant under multiplication with $\boldsymbol{\Gamma}$, and term $B$ equals $\mathbf{B}1$ if $\boldsymbol{\delta}$ is a probability distribution (i.e. if the elements of $\boldsymbol{\delta}$ sum to one). So, if these two requirements are fulfilled then $\boldsymbol{\delta}$ is the stationary distribution related to $\boldsymbol{\Gamma}$.

## b)

```
# Chapter 1, R-code for exercise 8, mwp 21/1-2011

# Calculate the stationary distribution using formula on p. 19
statdist <- function(gamma){
  m = dim(gamma)[1]
  matrix(1,1,m) %*% solve(diag(1,m) - gamma + matrix(1,m,m))
}
```

## c)

The results for the five cases are shown below:

```
# i
> statdist(i)
          [,1]      [,2]      [,3]
[1,] 0.4761905 0.2380952 0.2857143

# ii
> statdist(ii)
          [,1] [,2]      [,3]
[1,] 0.1666667  0.5 0.3333333

# iii
> statdist(iii)
          [,1]      [,2]      [,3]      [,4]
[1,] 0.3235294 0.2941176 0.1764706 0.2058824

# iv
> statdist(iv)
     [,1]          [,2] [,3] [,4]
```

```
[1,]    0 -5.551115e-17  0.4  0.6

# v
> statdist(v)
Error in solve.default(diag(1, m) - gamma + matrix(1, m, m)) :
  Lapack routine dgesv: system is exactly singular
```

Note for case iv that state 1 and 2 are transient, i.e. they are assigned zero probability in the stationary distribution. States that are not transient are called persistent (or recurrent). No stationary distribution exists in case v since the chain has two absorbing states (state 1 and state 4), i.e. states that the chain can never leave.

## Exercise 10

Define $\mathbf{B}v = (1, 2, \ldots, m)$ and $\mathbf{B}V = \text{diag}(1, 2, \ldots, m)$. Recall that $\gamma_{ij} = \Pr(C_{t+1}|C_t)$, $\mathbf{\Gamma} = \{\gamma_{ij}\}$, and that $\delta_i = \Pr(C_t = i)$ for all $t$ if the Markov chain is stationary. We have

$$\text{Cov}(C_t, C_{t+k}) = \text{E}[C_t C_{t+k}] - \text{E}[C_t]\text{E}[C_{t+k}].$$

The second term is

$$\begin{aligned}
\text{E}[C_t]\text{E}[C_{t+k}] &= \left[\sum_{i=1}^{m} i\Pr(C_t = i)\right]\left[\sum_{j=1}^{m} i\Pr(C_{t+k} = j)\right] \\
&= \boldsymbol{\delta}\mathbf{B}v'\boldsymbol{\delta}\mathbf{B}v' \\
&= (\boldsymbol{\delta}\mathbf{B}v')^2.
\end{aligned}$$

The first term is

$$\begin{aligned}
\text{E}[C_t C_{t+k}] &= \sum_{i=1}^{m}\sum_{j=1}^{m} ij\Pr(C_t = i, C_{t+k} = j) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} ij\Pr(C_{t+k} = j|C_t = i)\Pr(C_t = i) \\
&= \sum_{i=1}^{m} i\Pr(C_t = i)\sum_{j=1}^{m} j\Pr(C_{t+k} = j|C_t = i) \\
&= \sum_{i=1}^{m} i\Pr(C_t = i)\mathbf{\Gamma}^k\mathbf{B}v' \\
&= \boldsymbol{\delta}\mathbf{B}V\mathbf{\Gamma}^k\mathbf{B}v'.
\end{aligned}$$

Thus

$$\text{Cov}(C_t, C_{t+k}) = \boldsymbol{\delta}\mathbf{B}V\mathbf{\Gamma}^k\mathbf{B}v' - (\boldsymbol{\delta}\mathbf{B}v')^2.$$