Solution to exercises, week 1, principles of statistical estimation

1 MLE for the parameter of an exponential distribution

Density: $f(x) = \alpha \exp(-\alpha x)\mathrm{I}_{[0,+\infty)}(x)$

Likelihood: $L(x_1, ..., x_n; \alpha) = \prod_{i=1}^{n} \alpha \exp$

$(-\alpha x_i)\mathrm{I}_{[0,+\infty)}(x_i)$ Log-likelihood: $l(x_1, ..., x_n; \alpha) = l \ln \alpha - \alpha \sum_{i=1}^{n} x_i$

Derivating l w.r.t α, we get: $l'(\alpha) = n/\alpha - \sum_i x_i$ i.e. $\alpha = n/\sum_i x_i$

It is straighforward to verify that $\alpha = n/\sum_i x_i$ is a global maximum of l, hence of L.

Conclusion: $\hat{\alpha}_{ML} = n/\sum_i x_i$

Obtaining an expression of the bias analytically is not straightforward. In contrast, $1/\hat{\alpha}_{ML}$ is the emprical mean and is therefore an unbiased estimator of $1/\alpha$.

2 MLE for the parameter of a continuous uniform distribution

Density: $f(x) = \frac{1}{a}\mathrm{I}_{[0,+a)}(x)$

Likelihood: $L(x_1, ..., x_n; a) = \prod_{i=1}^{n} \frac{1}{a}\mathrm{I}_{[0,+a)}(x_i)$

Since we focus on a, it is convenient to rewite L as:

$L(x_1, ..., x_n; a) = \frac{1}{a^n}\mathrm{I}_{[0,M]}(a)$ our M is defined as $\max(x_1, ..., x_n)$. The expression is maximised (in a) when $a = M$. Hence $\hat{a}_{ML} = M$.

To obtain the bias of $\hat{a}$, it can be convenient to have an expression of its pdf. To this end, we first obtain the expression of the cdf of $\hat{a}$.

$$\begin{aligned} P(\hat{a} \leq u) &= P(M \leq u) && (1) \\ &= P(X_1 \leq u, ..., X_n \leq u) && (2) \\ &= \prod_i P(X_i \leq u) && (3) \\ &= \prod_i F(u) = F^n(u) && (4) \end{aligned}$$

Where F is the cdf of X. The cdf of M is the cdf of X to the power n. The pdf g of M is by definition the derivative of its cdf. We have therefore:

$$g(u) = nf(u)F^{n-1}(u) \qquad (5)$$

where f is the pdf of X.

$$g(u) = n\frac{1}{a}\mathrm{I}_{[0,+a)}(u)(\frac{u}{a})^{n-1} \qquad (6)$$

By definition the expectation of M is

$$\begin{aligned} E[\hat{a}] = E[M] &= \int ug(u)du && (7) \\ &= \int nu\frac{1}{a}\mathrm{I}_{[0,+a)}(u)(\frac{u}{a})^{n-1}du && (8) \end{aligned}$$

$$= \frac{n}{a^n}\left[\frac{u^{n+1}}{n+1}\right]_0^a \tag{9}$$

$$= a\frac{n}{n+1} \tag{10}$$

$E[\hat{a}] < a$ as it could be expected.

The bias is $E[\hat{a}] - a = a\frac{n}{n+1} - a = -a/(n+1)$ and tends to 0 when $n \longrightarrow +\infty$

3 Method of moments for the parameter of a continuous uniform distribution

In this problem, the expectation of a random varaible under the density considered is $a/2$. Hence we gave $a = 2E(X)/$ It makes sense to estimate a as $\hat{a}_{MM} = 2\bar{x} = 2\sum_i x_i/n$. It is straightforward to see that $\hat{a}_{MM}$ is unbiased. Its variance and MSE are therefore equal and it is more convenient to consider the variance:

$V[\hat{a}_{MM}] = V[2\sum\limits_i x_i/n] = \frac{4}{n^2}\sum\limits_i V[X_i] = \frac{4}{n}V[X]$.

The variance of a $U([0,a])$ distribution is $a/12$ (cf wikipedia).

We have therefore: $V[\hat{a}_{MM}] = a/(3n)$.

This estimator is unbiased for any n and increasingly accurate as n increases. It may look like a good choice. However, in presence of a single large outlier, it may return a value that is smaller than the largest sample value, which is a non-sense under this model.