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Solution to exercise 58

Question 1: With k factors each having p levels measured r times the total number of single measurements is

$$N = r \cdot p^k$$

In particular

$$(k, p, r) = (6, 2, 2) \Longrightarrow N = 2 \cdot 2^6 = 128$$
$$(k, p, r) = (6, 3, 2) \Longrightarrow N = 2 \cdot 3^6 = 1458$$

Question 2: A fractional factorial design is wanted and it is of the type p^{k-q}, where $p = 2$ and $k = 6$. Estimation of 6 main effects and a common level uses a total of $6 + 1 = 7$ degrees of freedom, so that the smallest possible design (of the type 2^{6-q}) is obviously with 8 single experiments, that is

$$\min_q (2^{6-q}) \geq 7 \Longrightarrow q = 3$$

i.e. a 2^{6-3} design.

The factors D, E and F are introduced in the complete underlying factorial structure defined by the factors A, B and C. We could use the generator equations

I		
A		
B		
AB		
C		
AC	$=$	E
BC	$=$	F
ABC	$=$	D

The factors (A, B, C, D, E, F) are given indices (i, j, k, l, m, n), respectively.

The design can be constructed by the tabular method or by the general Kempthorne's method. The tabular method yields:

A	B	C	D=ABC	E=-AC	F=-BC	
i	j	k	+ijk	-ik	-jk	Code
-1	-1	-1	-1	-1	-1	(1)
+1	-1	-1	+1	+1	-1	a de
-1	+1	-1	+1	-1	+1	b df
+1	+1	-1	-1	+1	+1	ab ef
-1	-1	+1	+1	+1	+1	c def
+1	-1	+1	-1	-1	+1	ac f
-1	+1	+1	-1	+1	-1	bc e
+1	+1	+1	+1	-1	-1	abc d

Using the Kempthorne method (solving index equations) gives as follows:

For the principale fraction the following index equations will apply:

$$(i+j+k+l)_2 = 0 \quad , \quad (i+k+m)_2 = 0 \quad , \quad (j+k+n)_2 = 0$$

Solutions are (3 linearly independent solutions are needed):

i	j	k	$\Rightarrow$	l	m	n	$navn$
1	0	0	$\Rightarrow$	1	1	0	$ade = x$
0	1	0	$\Rightarrow$	1	0	1	$bdf = y$
0	0	1	$\Rightarrow$	1	1	1	$cdef = z$

The principal fraction is then

$$\boxed{\begin{array}{cccc} (1) & x & y & xy \\ z & xz & yz & xyz \end{array}} = \boxed{\begin{array}{cccc} (1) & ade & bdf & abef \\ cdef & acf & bce & abcd \end{array}}$$

The defining relation for the design is

$$I = ABCD = ACE = BDE = BCF = ADF = ABEF = CDEF$$

and it is seen to have **resolution=III**.

Contrasts for the main effects for A and F are respectively:

$$[A] = -(1) + ade - bdf + abef - cdef + acf - bce + abcd$$
$$[F] = -(1) - ade + bdf + abef + cdef + acf - bce - abcd$$

Effect estimates (difference in response when a factor is changed from “low” to “high”)

$$\widehat{A} = \widehat{A}_1 - \widehat{A}_0 = 2\widehat{A}_1 = [A]/(2^{3-1})$$
$$\widehat{F} = \widehat{F}_1 - \widehat{F}_0 = 2\widehat{F}_1 = [F]/(2^{3-1})$$

Sums of squares are

$$\mathrm{SSQ}_A = \frac{[A]^2}{2^{6-3}} \text{ and } \mathrm{SSQ}_F = \frac{[F]^2}{2^{6-3}}$$

Question 3: 6 main effects are to be estimated, and they all require 3-1=2 degrees of freedom. The common level, μ, requires 1 degree of freedom. Thus the smallest possible design needs $6 \times 2 + 1 = 13$ measurements.

Thus the smallest possible design of type 3^{6-q} is the with 27 single measurements. So

$$\min_q(3^{6-q}) \geq 13 \Longrightarrow q = 3$$

i.e. a 3^{6-3} fractional fractorial design.

The introduction of the factors D, E and F in the complete factorial defined by A, B and C can be done in numerous ways. One set of generator equations with corresponding index equation for the principal fraction could be

I			
A			
B			
AB			
AB^2			
C			
AC			
AC^2	$=$	F	$\Rightarrow (i+2j)_3 = n$
BC			
BC^2	$=$	E	$\Rightarrow (j+2k)_3 = m$
ABC	$=$	D	$\Rightarrow (i+j+k)_3 = l$
ABC^2			
AB^2C			
AB^2C^2			

Theoretically 13 additively action factors (because $13 \times (3-1) + 1 = 27$) could be handled in a 3^3 design, that is the factors A, B and C plus further 10 factors (allthough this is a rather dangerous design since it relies on complete additivity, and no measures of interaction or departure from additivity can be computed).

Question 4: An adequate design divides the 27 single measurements in 3 blocks each containing 9 measurements. In order to do so yet another effect is chosen to be confounded with blocks. Again there are many possibilities, that could be considered, and in principle all interactions not allready used could be used.

If AB^2C^2 is chosen the block number is determined from the index of this effect, that is:

$$Block = AB^2C^2 \Longleftrightarrow \text{block no.} = (i+2j+2k)_3$$

Question 5: 16 single measurements are needed. Thus 2 blocks each containing 8 measurements will do:

First design :

1st replicate

(1)	ade	bdf	$abef$
$cdef$	acf	bce	$abcd$

2nd replicate

(1)	ade	bdf	$abef$
$cdef$	acf	bce	$abcd$

Complete randomisation **within** the two blocks is employed.

If the single measurements are to be repeated within the blocks one could choose to confound yet another suitable effect with blocks, for example as:

I		
A		
B		
AB	$=$	$block$
C		
AC	$=$	E
BC	$=$	F
ABC	$=$	D

In this way the block number is given by $(i+j)_2$ and the division in blocks is:

Second design:

Block no. 0

(1)	(1)	$abef$	$abef$
$cdef$	$cdef$	$abcd$	$abcd$

Block no. 1

ade	ade	bdf	bdf
acf	acf	bce	bce

Complete randomisation **within** the two blocks is employed.

The essential difference on the two designs is, that in the second design the block effect (which corresponds to difference between the the two replications) will also be confounded with AB.

So generally one would choose the first design.

Uncertainty and tests of effects will be practically the same. One minor difference is that in the second design the residual sum of squares will have 8 degrees of freedom, but only 7 in the first.

Finally, if it is affordable to use 16 measurements it would perhaps be more sensible to use a 2^{6-2} design in two blocks with underlying complete factorial defined by A, B, C and D.

The generator equations could then be chosen as $F = BCD$, $E = ACD$ and block$=ABCD$, for example. The design then becomes:

Block no. 0

(1)	$abef$	acf	cd
bce	adf	bde	$abcdef$

Block no. 1

a	bef	cf	acd
$abce$	df	$abde$	$bcdef$

This results in a resolution IV design in contrast to the previous resolution III design.

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