IMM - DTU

Statistical Design and Analysis of Experiments
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Solution to exercise 52

Question 1:

Construction		
I		
A		
B		
AB		
C		
AC	=	D
BC		
ABC	=	E

$\Longrightarrow$

Alias relations with I_1=ABCE and I_2=ACD						
I	=	ABCE	=	ACD	=	BDE
A	=	BCE	=	CD	=	ABDE
B	=	ACE	=	ABCD	=	DE
AB	=	CE	=	BCD	=	ADE
C	=	ABE	=	AD	=	BCDE
AC	=	BE	=	D	=	ABCDE
BC	=	AE	=	ABD	=	CDE
ABC	=	E	=	BD	=	ACDE

Note: Another, possibly more natural, choice could be D=AC and E=BC which will give the defining relation
I=ACD=BCE=ABDE
which also will give a resolution III design.

The single experiment "(1)" corresponds to A=10%, B=60^{o}C, C=no, D=4% and E=dry, so it is the principal fraction that is asked for.

The principal fraction consists of all single experiments with an **even** number of letters in common with ABCE and ACD:

(1)	ade	be	abd
cde	ac	bcd	abce

No main effects or the interaction AB are confounded with each other such that it under the given assumptions is possible to estimate and test all relevant effects. An alternative is to choose D=BC instead of D=AC. The design is a resolution III design.

Using the 'tabular' method (recommended) gives, if D=-AC and E=+ABC are chosen:

A	B	C	D = -AC	E=ABC	Code
-1	-1	-1	-1	-1	(1)
+1	-1	-1	+1	+1	ade
-1	+1	-1	-1	+1	be
+1	+1	-1	+1	-1	abd
-1	-1	+1	+1	+1	cde
+1	-1	+1	-1	-1	ac
-1	+1	+1	+1	-1	bcd
+1	+1	+1	-1	+1	abce

i.e. the principal fraction.

The problem can also be solved using Kempthorne's method, i.e. the general method for p^k factorial designs. Indices are i, j, k, l, m for the factors A, B, C, D and E, respectively, which gives:

$$I_1\text{=ABCE} \Leftrightarrow i + j + k + m = 0 \quad \text{and} \quad I_2\text{=ACD} \Leftrightarrow i + k + l = 0$$

We need 3 linearly independent solutions since then principal fraction has size 2^3.

$$(i=1, j=0, k=0) \Longrightarrow (l=1, m=1) \text{ , i.e. x=ade}$$

$$(i=0, j=1, k=0) \Longrightarrow (l=0, m=1) \text{ , i.e. y=be}$$

$$(i=0, j=0, k=1) \Longrightarrow (l=1, m=1) \text{ , i.e. z=cde}$$

The gives the design wanted:

(1)	x	y	xy
z	xz	yz	xyz

$\Longrightarrow$

(1)	ade	be	abd
cde	ac	bcd	abce

Question 2:

Construction		
I		
A		
B		
AB		
C		
AC	=	D
BC	=	block
ABC	=	E

All single experiments with an **even** number of letters in common with BC go in one block and the ones with an **uneven** ' number of letters go in the other block:

Block = BC

Block 0		Block 1	
(1)	ade	be	abd
bcd	abce	cde	ac

Using the tabular method yields if we again apply D=-AC and E=+ABC and block=BC :

A	B	C	D = -AC	E=ABC	block=BC	Code
-1	-1	-1	-1	-1	+1	(1)
+1	-1	-1	+1	+1	+1	ade
-1	+1	-1	-1	+1	-1	be
+1	+1	-1	+1	-1	-1	abd
-1	-1	+1	+1	+1	-1	cde
+1	-1	+1	-1	-1	-1	ac
-1	+1	+1	+1	-1	+1	bcd
+1	+1	+1	-1	+1	+1	abce

which again is the principal fraction, but now distributed on two blocks (+1 and -1).

One can also solve the equations (Kempthorne's method) $j+k=0$ (block 0) and $j+k=1$:

$$(i=1, j=0) \Longrightarrow (k=0, l=1, m=1) \text{ , i.e. x=ade}$$

$$(i=0, j=1) \Longrightarrow (k=1, l=1, m=0) \text{ , i.e. y=bcd}$$

and the four single experiments are (1), x, y and xy, i.e. (1), ade, bcd and abce.

The other block can be found by e.g. finding one solution to $j+k=1$ together with the previous equations, that is:

$$i+j+k+m=0 \ \ , \ \ i+k+l=0 \ \text{ and } \ j+k=1$$

$$(i=1, j=0) \Longrightarrow (k=1, l=0, m=0) \text{ , i.e. x=ac ,}$$

that is multiplied on the first (principal) block. This gives the four single experiments ac, cde, abd and be.

It is noted that all effects in the alias relation corresponding to BC will be confounded with each other as well as with blocks.

Question 3:

If the factor D is introduced into the complete factorial for the factors A, B and C by the relation D=ABC, the defining relation for the design will be I=ABCD.

The principal fraction for this design consists of all single experiments that have an **even** number of letters in common with ABCD, and they are precisely those given in the text.

The 'tabular' method can also be applied.

The design is a resolution IV design.

Alias relations		
I	=	ABCD
A	=	BCD
B	=	ACD
AB	=	CD
C	=	ABD
AC	=	BD
BC	=	AD
ABC	=	D

If Kempthorne's method is used we have to solve the index equation $i+j+k+l=0$, and three linearly independent solutions are needed. These solutions could be $(i,j,k,l) = (1,0,0,1)$, $(0,1,0,1)$ and $(0,0,1,1)$, corresponding to x=ad, y=bd and z=cd, from which the design given is derived as (1), x, y, xy, z, xz, yz, xyz.

Question 4:

The standard sequence corresponding to the underlying factorial (A,B,C) is used together with Yates' algorithm:

Response			Yates' algorithm			SSQ	Estimates		
(1)	=	60	130	246	538	36180.5	$\widehat{\mu}$	=	67.25
a d	=	70	116	292	26	84.5	$\widehat{A}_1$	=	3.25
b d	=	55	152	16	–26	84.5	$\widehat{B}_1$	=	–3.25
ab	=	61	140	10	38	180.5	$\widehat{AB}_{11}$	=	4.75
c d	=	84	10	–14	46	264.5	$\widehat{C}_1$	=	5.75
ac	=	68	6	–12	–6	4.5	$\widehat{AC}_{11}$	=	–0.75
bc	=	57	–16	–4	2	0.5	$\widehat{BC}_{11}$	=	0.25
abc d	=	83	26	42	46	264.5	$\widehat{ABC}_{111} = \pm\widehat{D}_1$	=	5.75

All sums of squares have 1 degree of freedom. The sign of the D–estimate can be found by considering a single experiment fx (1):

Indices in experiment "(1)"

A	B	C	D
0	0	0	0

$\Longrightarrow$ $ABC_{000}=+D_0 \Longrightarrow ABC_{111}=+D_1$

resulting in $\widehat{D}_1 = +\ 5.75$.

If all interactions with D and the three factor interaction ABC are removed we get the following confoundings (namely none!):

Confoundings		
I	=	
A	=	
B	=	
AB	=	
C	=	
AC	=	
BC	=	
	=	D

Testing can be carried out by means of the $\widehat{\sigma}^2 = 4.84$ with 4 degrees of freedom given in the text. Thereby critical F(1,4)–values for all estimates can be found. The critical F–value could be F=F$(1,4)_{0.95}$=7.71 in testing at an $\alpha = 5\%$ level.

All SSQ's, that are larger than 4.84·7.71=37.32 will thereby be significant. This will apply for the effects for A, B, AB, C and D, but not for AC or BC (test quantity for D is appearing in the ABC–row in the Yates algorithm table).