

Solution to exercise 48

Question 1

S = plant = random effect: $S_i \in N(0, \sigma_s^2)$

B = evaluator = random effect: $B_j \in N(0, \sigma_B^2)$

l = alloy = deterministic effect: $\sum_{k=1}^5 l_k = 0$

The factor B is nested within S , and the mathematical model is therefore:

$$Y_{ijk r} = \mu + S_i + B(S)_{j(i)} + l_k + SL_{ik} + BL(S)_{jk(i)} + E_{r(ijk)}$$

Computation of EMS table: effects and components of variance

	3	4	5	1	σ_S^2	$\sigma_{B(S)}^2$	ϕ_l	σ_{SL}^2	$\sigma_{BL(S)}^2$	σ_E^2
	i	j	k	r						
S_i	1	4	5	1	20	5	—	4	1	1
$B(S)_{j(i)}$	1	1	5	1		5	—	—	1	1
l_k	3	4	0	1			12	4	1	1
SL_{ik}	1	4	1	1				4	1	1
$BL(S)_{jk(i)}$	1	1	1	1					1	1
$E_{r(ijk)}$	1	1	1	1						1

The table indicates the possible test that can be constructed and which effects that cannot be tested without further assumptions.

The main effect l_k can be tested against the SL -effect or, if SL is not significant, against the $BL(S)$ -effect.

Question 2

From the data the sums of squares can be computed as (given in the text also):

$$\begin{aligned} \text{SSQ}_{tot} &= 11102 - \frac{614656}{60} = 857.73; & df &= 59 \\ \text{SSQ}_S &= \frac{209070}{20} - \frac{614656}{60} = 209.23; & df &= 2 \\ \text{SSQ}_B &= \frac{154110}{15} - \frac{614656}{60} = 29.73; & df &= 3 \\ \text{SSQ}_l &= \frac{128200}{12} - \frac{614656}{60} = 439.06; & df &= 4 \\ \text{SSQ}_{SL} &= \frac{43690}{4} - \frac{614656}{60} - \text{SSQ}_l - \text{SSQ}_S = 29.94; & df &= 8 \end{aligned}$$

$$\begin{aligned}
SSQ_{BL} &= \frac{32242}{3} - \frac{614656}{60} - SSQ_B - SSQ_l = 34.28; & df &= 12 \\
SSQ_{SB} &= \frac{52674}{5} - \frac{614656}{60} - SSQ_S - SSQ_B = 51.57; & df &= 6 \\
SSQ_{SBL} &= 857.73 - 209.23 - 29.73 - 439.06 \\
&= -29.94 - 34.28 - 51.57 = 63.92; & df &= 24
\end{aligned}$$

These sums of squares correspond to a complete crossed classification of the factors. For our model we find:

$$\begin{aligned}
SSQ_{B(S)} &= SSQ_B + SSQ_{BS} \\
&= 29.73 + 51.57 = 81.30; & df &= 3 + 6 = 9 \\
SSQ_{BL(S)} &= SSQ_{BL} + SSQ_{BLS} \\
&= 34.28 + 63.92 = 98.20; & df &= 12 + 24 = 36
\end{aligned}$$

This leads to the following ANOVA table:

Effect	SSQ	df	s^2	EMS
S	209.23	2	104.62	$20\sigma_S^2 + 5\sigma_{B(S)}^2 + 4\sigma_{SL}^2 + \sigma_{BL(S)}^2 + \sigma_E^2$
$B(S)$	81.30	9	9.03	$5\sigma_{B(S)}^2 + \sigma_{BL(S)}^2 + \sigma_E^2$
l	439.06	4	109.77	$12\phi_l + 4\sigma_{SL}^2 + \sigma_{BL(S)}^2 + \sigma_E^2$
SL	29.94	8	3.74	$4\sigma_{SL}^2 + \sigma_{BL(S)}^2 + \sigma_E^2$
$BL(S)$	98.20	36	2.73	$\sigma_{BL(S)}^2 + \sigma_E^2$
Residual	0.00	0	—	(no degrees of freedom) σ_E^2
Total	857.73	59		

Test of SL : $F = \frac{3.74}{2.73} = 1.37 \sim F(8, 36)_{0.75}$, which then is not significant.

We can now pool SL and $BL(S)$ and obtain an improved estimate for $\sigma_{BL(S)}^2 + \sigma_E^2$ with 44 degrees of freedom:

$$s_{pooled}^2 = (29.94 + 98.20)/(8 + 36) = 2.91; \quad df = 44$$

$$\text{Test of } l: F = 109.77/2.91 = 37.69 \gg F(4, 44)_{0.99}.$$

The alloy effect is therefore highly significant.

$$\text{Test of } B(S): F = 9.03/2.91 = 3.10 F(9, 44)_{0.99}$$

$$\text{Test of } S: F = 104.62/9.03 = 11.59 F(2, 9)_{0.99}$$

It is seen that both the $B(S)$ - and the S -effect are significant.

Question 3

Conclusion:

$$Y_{ijkl} = \mu + S_i + B(S)_{j(i)} + l_k + BL(S)_{jk(i)} + E_{r(ijk)}$$

($\sigma_{BL(S)}^2$ and σ_E^2 cannot be estimated separately.)

$$\begin{aligned}\hat{\sigma}_{BL(S)}^2 + \hat{\sigma}_E^2 &= 2.91 = 1.71^2 \\ \hat{\sigma}_S^2 &= \frac{1}{20} (s_S^2 - s_{B(S)}^2) = (104.62 - 9.03)/20 = 4.78 = 2.19^2 \\ \hat{\sigma}_{B(S)}^2 &= \frac{1}{5} (s_{B(S)}^2 - s_{BL(S)}^2) = (9.03 - 2.91)/5 = 1.22 = 1.11^2 \\ \hat{\mu} &= 784/60 = 13.07 \\ \hat{l} &= \{195/12 - 13.07, 198/12 - 13.07, 219/12 - 13.07, \\ &\quad 133/12 - 13.07, 129/12 - 13.07\} \\ &= \{3.18, 3.43, -2.32, -1.98, -2.32\}\end{aligned}$$

Question 4

We can construct a reasonable *contrast*, namely $3(T_1 + T_2) - 2(T_3 + T_4 + T_5) = 397.00$. The sum of squares for this contrast is

$$SSQ_{contrast} = 397^2 / (12(9 + 9 + 4 + 4 + 4)) = 437.80$$

The *l*-SSQ can therefore be split into:

variation between alloys (1+2) against (3+4+5) = 437.80; $f = 1$

variation within alloys (1+2) and (3+4+5) = 439.06-437.80=1.26; $f = 1 + 2 = 3$

and it is seen that practically all the variation from alloys can be connected to the variation between the two groups while the variation within the groups is small.

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