IMM - DTU

Statistical Design and Analysis of Experiments
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Solution to exercise 46

Question 1:

We choose two defining contrasts (block confoundings) for example I_1=ABC and I_2=BCD, which will partition the design in 2×2 blocks. In doing this further $I_1 I_2$ = $AB^2C^2D \rightarrow$ AD will be confounded with blocks :

Construction		
I		
A		
B		
AB		
C		
AC		
BC		
ABC	=	I_1
D		
AD	=	$I_1 I_2$
BD		
ABD		
CD		
ACD		
BCD	=	I_2
ABCD		

If it is acceptable that AC is also confounded with blocks the design can be used. Other possibilities could be I_1=ABD and I_2=BCD etc.

The partitioning in blocks can be accomplished by using the "even/uneven" rule as shown in the following table:

	ABC/even		ABC/uneven	
BCD/even	(1)	acd	a	cd
	bc	abd	abc	bd
BCD/uneven	d	ac	b	abcd
	bcd	ab	ad	c

If the tabular method (recommended) is used we get:

A	B	C	D	I_1=ABC	I_2=BCD	Block no.	fx
-1	-1	-1	-1	-1	-1	1	(-1,-1)
+1	-1	-1	-1	+1	-1	2	(+1,-1)
-1	+1	-1	-1	+1	+1	4	(+1,+1)
+1	+1	-1	-1	-1	+1	3	(-1,+1)
-1	-1	+1	-1	+1	+1	4	
+1	-1	+1	-1	-1	+1	3	
-1	+1	+1	-1	-1	-1	1	
+1	+1	+1	-1	+1	-1	2	
-1	-1	-1	+1	-1	+1	2	
+1	-1	-1	+1	+1	+1	4	
-1	+1	-1	+1	+1	-1	2	
+1	+1	-1	+1	-1	-1	1	
-1	-1	+1	+1	+1	-1	2	
+1	-1	+1	+1	-1	-1	1	
-1	+1	+1	+1	-1	+1	3	
+1	+1	+1	+1	+1	+1	4	

Block 1 is (1), bc, abd, acd, that is the principal block.

Finally the problem can be solved using Kempthorne's method, i.e. the general method for p^k factorial designs.

Indices are chosen as i, j, k and l for the factors A, B, C og D, respectively, from which the principal block is constructed:

$$I_1 = ABC \Leftrightarrow i + j + k = 0 \quad \text{and} \quad I_2 = BCD \Leftrightarrow j + k + l = 0$$

We need two linearly independent solutions since the principal block is of size 2^2.

$$(i = 1, j = 0) \Longrightarrow (k = 1, l = 1) \text{ , i.e. } x = acd$$

$$(i = 0, j = 1) \Longrightarrow (k = 1, l = 0) \text{ , i.e. } y = bc$$

This gives the principal block:

(1)	x	$\Longrightarrow$	(1)	acd
y	xy		bc	abd

The remaining three blocks can be found as solutions to $(i+j+k = 1, j+k+l = 0)$, $(i+j+k = 0, j+k+l = 1)$ and $(i+j+k = 1, j+k+l = 1)$. One can also identify a single experiment in each of the three blocks and "multiply" the principal block with this experiment by which the three blocks will result. We may fx consider the single experiment "a":

a ×	(1)	acd	=	a	cd
	bc	abd		abc	bd

Question 2:

The simplest method is to introduce the factor D into the complete factorial composed by the factors A, B and C:

Construction				Alias relations		
I				I	=	ABCD
A				A	=	BCD
B				B	=	ACD
AB			$\Longrightarrow$	AB	=	CD
C				C	=	ABD
AC				AC	=	BD
BC				BC	=	AD
ABC	=	D		ABC	=	D

The underlined effects are thought to be zero (or very small) as explained in the text to the exercise.

All single experiments with an **uneven** number of letters in common with ABCD will appear in the principal fraction:

(1)	ad	bd	ab
cd	ac	bc	abcd

Alternatively one can choose the complementary fraction. It can be found by multiplying the principal fraction with "a" fx:

a ×	(1)	ad	bd	ab	=	a	d	abd	b
	cd	ac	bc	abcd		acd	c	abc	bcd

If the tabular method (recommended) is used we get (choosing fx D = +ABC):

A	B	C	D = ABC	code
-1	-1	-1	-1	(1)
+1	-1	-1	+1	ad
-1	+1	-1	+1	bd
+1	+1	-1	-1	ab
-1	-1	+1	+1	cd
+1	-1	+1	-1	ac
-1	+1	+1	-1	bc
+1	+1	+1	+1	abcd

i.e. the principal fraction (containing "(1)").

If the general Kempthorne's method is used, we have to solve the index equation $i+j+k+l=0$, and 3 linearly independent solutions are needed since the (two) fractions are of size 2^3. These solutions could be $(i,j,k,l)=(1,0,0,1)$, $(0,1,0,1)$ and $(0,0,1,1)$, which correspond to $x = ad$, $y = bd$ and $z = cd$, from which the individual experiments are derived as $(1), x, y, xy, z, xz, yz, xyz$.

Question 3:

Construction		
I		
A		
B		
AB		
C		
AC		
BC	=	blocks
ABC	=	D

By means of the "even/uneven" rule the experiment is partitioned in two blocks (and assuming it is the principal fraction we have chosen we get):

BC even		BC uneven	
(1)	ad	bd	ab
bc	abcd	cd	ac

Using the tabular method yields:

A	B	C	D = ABC	code	block=BC
-1	-1	-1	-1	(1)	+1
+1	-1	-1	+1	ad	+1
-1	+1	-1	+1	bd	-1
+1	+1	-1	-1	ab	-1
-1	-1	+1	+1	cd	-1
+1	-1	+1	-1	ac	-1
-1	+1	+1	-1	bc	+1
+1	+1	+1	+1	abcd	+1

i.e. the blocks (1) , ad , bc , abcd and (bd , ab , cd , ac .

If the general method is used one can construct the principal block by finding two solutions to the index equations $i+j+k+l=0$ and $j+k=0$. Two solutions are (1,0,0,1) $\Rightarrow$ x=ad and (0,1,1,0) $\Rightarrow$ y=bc, from which the block is derived as (1),x,y,xy. The other block is the solutions to $i+j+k+l=0$ and $j+k=1$, where it now suffices to find one solution and multiply that solution on the principal block.

The computations for the experiment can be carried by means of the underlying complete factorial defined by (A,B,C) and Yates' algorithm:

Responser		Yates' algorithm	SSQ	Estimates	
(1)	=			$\widehat{\mu}$	=
a d	=			$\widehat{A}_1$	=
b d	=			$\widehat{B}_1$	=
ab	=			$\widehat{AB}_{11}$	=
c d	=			$\widehat{C}_1$	=
ac	=			$\widehat{AC}_{11}$	=
bc	=			$\widehat{BC}_{11}$ = blocks	
abc d	=			$\widehat{ABC}_{111} = \pm\widehat{D}_1$	=

All sums of squares have one degree of freedom. The sign of the D-estimate can fx be found by considering one single experiment, fx (1):

Indices in experiment "(1)"

A	B	C	D
0	0	0	0

$\Longrightarrow$ ABC_{000}=+D_0 $\Longrightarrow$ ABC_{111}=+D_1

If all interactions involving D and the three factor interaction ABC are disregarded we get the following confoundings:

Confoundings		
I	=	
A	=	
B	=	
AB	=	
C	=	
AC	=	
BC	=	blocks
	=	D

The BC effect is confounded with blocks (and the AD effect), and in principle one can neither estimate nor test it. The sum of squares for for D appears in the ABC–row in the Yates table.