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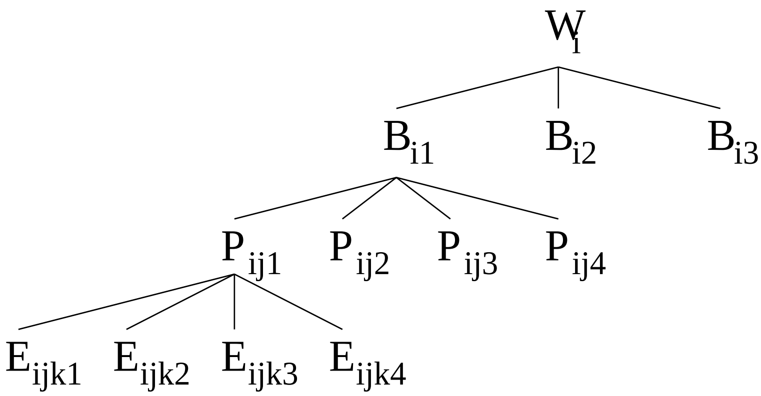
Solution to exercise 41, new version (2007)

Each week (W) 3 batches (B) are selected and from each batch samples are taken from 4 productions (P) (each batch gives rize to several productions). The conductivity is measured on 4 tiles (E) from each production. The structure of the data is thus (in principle):



in that each week has 3 batches, each batch has 4 productions and each production has 4 tiles, that is a complete hierarchial structure which can be modelled using the terms:

$$B(W) \;\; , \;\; P(WB) \;\; \text{and} \;\; E(WBP)$$

Question 1
We are interested in identifying sources of variation and it is therefore reasonable to model all effects as random. The statistical model is then:

$$Y_{ijkl} = \mu + W_i + B(W)_{j(i)} + P(WB)_{k(ij)} + E_{l(ijk)},$$

where μ is the general level and

$$W_i \in N(0, \sigma^2_W); \qquad B(W)_{j(i)} \in N(0, \sigma^2_{B(W)});$$
$$P(WB)_{k(ij)} \in N(0, \sigma^2_{P(WB)}); \qquad E_{l(ijk)} \in N(0, \sigma^2_E)$$

For brevity the term $E(WBP)_{l(ijk)}$ is written as $E_{l(ijk)}$.

EMS-table for ANOVA

	i	j	k	l	σ^2_W	$\sigma^2_{B(W)}$	$\sigma^2_{P(WB)}$	σ^2_E
	3	3	4	4				
W_i	1	3	4	4	48	16	4	1
$B(W)_{j(i)}$	1	1	4	4		16	4	1
$P(WB)_{k(ij)}$	1	1	1	4			4	1
$E_{l(ijk)}$	1	1	1	1				1

The $P(WB)$ effect is tested against the E effect. The $B(W)$ effect is tested against the $P(WB)$ effect. The W effect is tested against the $B(W)$ effect.

If reasonable (after testing) sums of squares from not significant effects can be pooled which will lead to stronger tests.

If, for example, the P(WB) effect is not significant its SSQ can be pooled with the residual SSQ and subsequently the $B(W)$ term can be tested against the new residual SSQ.

Question 2

The sums of squares for our model can be derived from the sums of squares for a completely crossed splitting of the variation:

$$\begin{aligned}
\mathrm{SSQ}_W &= 411.3,\ f=2 \\
\mathrm{SSQ}_{B(W)} &= \mathrm{SSQ}_B + \mathrm{SSQ}_{WB} = 211.1,\ f=6 \\
\mathrm{SSQ}_{P(WB)} &= \mathrm{SSQ}_P + \mathrm{SSQ}_{BP} + \mathrm{SSQ}_{WP} + \mathrm{SSQ}_{WBP} = 545.9,\ f=27 \\
\mathrm{SSQ}_{E(WBP)} &= \mathrm{SSQ}_E + \mathrm{SSQ}_{EW} + \mathrm{SSQ}_{EB} + \mathrm{SSQ}_{EWB} + \mathrm{SSQ}_{EP} + \\
&\quad \mathrm{SSQ}_{EWP} + \mathrm{SSQ}_{EBP} + \mathrm{SSQ}_{EWBP} = 180.1,\ f=108.
\end{aligned}$$

The ANOVA table thus becomes:

Variation	SSQ	f	EMS
W	411.3	2	$48\sigma^2_W + 16\sigma^2_{B(W)} + 4\sigma^2_{P(WB)} + \sigma^2_E$
$B(W)$	211.1	6	$16\sigma^2_{B(W)} + 4\sigma^2_{P(WB)} + \sigma^2_E$
$P(WB)$	545.9	27	$4\sigma^2_{P(WB)} + \sigma^2_E$
Residual E	180.1	108	σ^2_E
Total	1348.4	143	

$H_0^{(1)} : \sigma^2_{P(WB)} = 0$ against $H_1^{(1)} : \sigma^2_{P(WB)} \neq 0$

$F = \frac{545.9/27}{180.1/108} = 12.1,$

which is compared with an $F(27, 108)$-distribution. It is strongly significant.

$H_0^{(2)}: \sigma^2_{B(W)} = 0$ against $H_1^{(2)}: \sigma^2_{B(W)} \neq 0$

$F = \frac{211.1/6}{545.9/27} = 1.7 \sim F(6,27)_{0.16} \Rightarrow H_0^{(2)}$.

New pooled $\text{SSQ}_{P(WB)} = 191.3 + 535.7 = 757.0,\ f = 33$.

$H_0^{(3)}: \sigma^2_W = 0$ against $H_1^{(3)}: \sigma^2_W \neq 0$

$F = \frac{411.3/2}{757.0/33} = 9.0 > F(2,33)_{0.001} \Rightarrow H_1^{(3)}$

Thus the identified model is:

$$Y_{ijkl} = \mu + W_i + P(WB)_{k(ij)} + E_{l(ijk)}$$

which includes a variation between weeks and between productions, while batches do not seem to be very different.

Question 3

The estimates corresponding to the final model are:

$$\begin{aligned}
\hat{\mu} &= \overline{x} \\
\hat{\sigma}^2_E &= \frac{180.1}{108} = 1.667 = 1.29^2 \\
\hat{\sigma}^2_{P(WB)} &= \frac{1}{4}\left(\frac{757.0}{33} - \frac{180.1}{108}\right) = 5.318 = 2.31^2 \\
\hat{\sigma}^2_W &= \frac{1}{48}\left(\frac{411.3}{2} - \frac{757.0}{33}\right) = 3.806 = 1.95^2
\end{aligned}$$

One could removed the batches from the model and conclude:

$$Y_{ijkl} = \mu + W_i + P(W)_{k(i)} + E_{l(ik)}$$

where now k is production number within one week and then

$$\begin{aligned}
\hat{\mu} &= \overline{x} \\
\hat{\sigma}^2_E &= \frac{180.1}{108} = 1.667 = 1.29^2 \\
\hat{\sigma}^2_{P(W)} &= \frac{1}{4}\left(\frac{757.0}{33} - \frac{180.1}{108}\right) = 5.318 = 2.31^2 \\
\hat{\sigma}^2_W &= \frac{1}{48}\left(\frac{411.3}{2} - \frac{757.0}{33}\right) = 3.806 = 1.95^2
\end{aligned}$$

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