IMM - DTU
Statistical Design and Analysis of Experiments
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Solution to Montgomery exercise 15-17, 6th ed. (14-16, 5th ed.)

The problem is an analysis of variance with a covariate x = hardness.

$$Y_{ij} = \mu + \tau_i + \beta x_{ij} + E_{ij}; \quad i = 1,2,3, \; j = 1,\ldots,5, \; \tau_1 + \tau_2 + \tau_3 = 0$$

Cutting speed					
1000		1200		1400	
y	x	y	x	y	x
68	120	112	165	118	175
90	140	94	140	82	132
98	150	65	120	73	124
77	125	74	125	92	141
88	136	85	133	80	130
421	671	430	683	445	702

$$\begin{aligned}
S_{yy} &= 68^2 + 90^2 + 98^2 + \cdots + 92^2 + 80^2 \\
&\quad - \frac{(421+430+445)^2}{3\cdot 5} = 115148 - 1296^2/15 = 3173.6 \\
S_{xx} &= 120^2 + 140^2 + 150^2 + \cdots + 141^2 + 130^2 \\
&\quad - \frac{(671+683+702)^2}{3\cdot 5} = 285366 - 2056^2/15 = 3556.93 \\
S_{xy} &= 68\cdot 120 + 90\cdot 140 + 98\cdot 150 + \cdots + 80\cdot 130 \\
&\quad - \frac{1296\cdot 2056}{3\cdot 5} = 180946 - 177638.4 = 3307.6
\end{aligned}$$

$$\begin{aligned}
T_{yy} &= \frac{421^2 + 430^2 + 445^2}{5} - \frac{1296^2}{3\cdot 5} = 58.8 \\
T_{xx} &= \frac{671^2 + 683^2 + 702^2}{5} - \frac{2056^2}{3\cdot 5} = 97.73 \\
T_{xy} &= \frac{421\cdot 671 + 430\cdot 683 + 445\cdot 702}{5} - \frac{1296\cdot 2056}{3\cdot 5} = 75.8
\end{aligned}$$

$$\begin{aligned}
E_{yy} &= S_{yy} - T_{yy} = 3114.8 \\
E_{xx} &= S_{xx} - T_{xx} = 3459.2 \\
E_{xy} &= S_{xy} - T_{xy} = 3231.8 \\
SS_E &= E_{yy} - \frac{E_{xy}^2}{E_{xx}} = 95.45 \\
SS'_E &= S_{yy} - \frac{S_{xy}^2}{S_{xx}} = 97.85
\end{aligned}$$

Model alternatives	Residual SSQ	d.f.
$Y_{ij} = \mu + \tau_i + \beta x_{ij} + E_{ij}$	$SS_E = 95.45$	15 - 1 - (3-1) - 1
$Y_{ij} = \mu + \beta x_{ij} + E_{ij}$	$SS'_E = 97.85$	15 - 1 - 1
$Y_{ij} = \mu + \tau_i + E_{ij}$	$E_{yy} = 3114.8$	15 - 1 - (3-1)
$Y_{ij} = \mu + E_{ij}$	$S_{yy} = 3173.6$	15 - 1

We can now compute the test quantities for cutting speed (the treatment) and the hardness (the covariate):

$F_{Cuttingspeed} = \frac{(SS'_E - SS_E)/(3-1)}{SS_E/11} = \frac{2.40/2}{95.45/11} = 0.14$

$F_{covariate} = \frac{(E_{yy} - SS_E)/(1)}{SS_E/11} = \frac{(3114.8-95.45)/1}{95.45/11} = \frac{3019.35/1}{95.45/11} = 347.90$

The textbook computes the above quantities according to the following table (which I do not find particularly useful):

Source of variation	x	xy	y	y	Degrees of freedom	s^2
Cutting speed	97.73	75.8	58.8			
Error	3459.2	3231.8	3114.3	95.45	11	8.68
Total	3556.9	3307.9	3173.6	97.85	13	
Adjusted SSQ				2.40	2	1.20

The model can be reduced to: $Y_{ij} = \mu_0 + \beta_0 \cdot x_{ij} + E_{ij}$

$$\begin{aligned} \widehat{\beta}_0 &= S_{xy}/S_{xx} = 3307.6/3556.93 = 0.93 \\ \widehat{\mu}_0 &= \overline{Y} - \widehat{\beta}_0 \cdot \overline{x} = \frac{1296}{15} - 0.93 \cdot \frac{2056}{15} = -41.06 \end{aligned}$$

New residual variation: $SSQ_{res} = 97.85$ which is the residual variance for the model without the treatment with 15-1-1=13 degrees of freedom.

Estimate for uncertainty variance: $\widehat{\sigma}_E^2 = 97.85/13 = 7.53 \simeq 2.7^2$.

The problem may also be formulated as a general linear regression model in matrix formulation:

$$\underline{Y} = \underline{\underline{x}} \cdot \underline{\theta} + \underline{E}$$

$$\left\{ \begin{array}{c} 68 \\ 90 \\ 98 \\ 77 \\ \vdots \\ 92 \\ 80 \end{array} \right\} = \left\{ \begin{array}{ccccc} 1 & 1 & 0 & 0 & 120 \\ 1 & 1 & 0 & 0 & 140 \\ 1 & 1 & 0 & 0 & 150 \\ 1 & 1 & 0 & 0 & 125 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & 1 & 141 \\ 1 & 0 & 0 & 1 & 130 \end{array} \right\} \cdot \left\{ \begin{array}{c} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \beta \end{array} \right\} + \left\{ \begin{array}{c} E_{11} \\ \vdots \\ \vdots \\ \vdots \\ E_{35} \end{array} \right\}$$

The estimate for $\underline{\theta}$ can in principle be found by solving the normal equations $(\underline{\underline{x}}^T \underline{\underline{x}})\hat{\underline{\theta}} = \underline{\underline{x}}^T \underline{Y}$. By estimating alternative versions of this model its terms can be tested - see above.