Statistical Design and Analysis of Experiments
Part Two

Lecture notes

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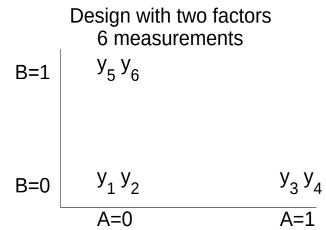
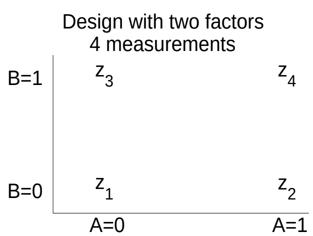
Factorial experiments - introduction



The estimate of the A-effect based on y:

$$\widehat{A}_y = [(y_3 + y_4) - (y_1 + y_2)]/2$$



The estimate of the A-effect based on z:

$$\widehat{A}_z = [(z_2 + z_4) - (z_1 + z_3)]/2$$

One-factor-at-the-time or factorial design

Are $\widehat{A}_y$ and $\widehat{A}_z$ equivalent ?

Var$\widehat{A}_y$ = ?

Var$\widehat{A}_z$ = ?

Additive model:

$$Response = \mu + A + B + residual$$

Can it always be applied?

More complicated model:

$$Response = \mu + A + B + AB + residual$$

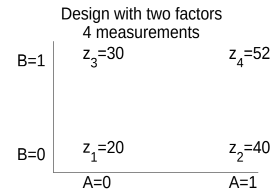
Is it more needed for factorial designs than for block designs, for example, where additivity is often assumed?

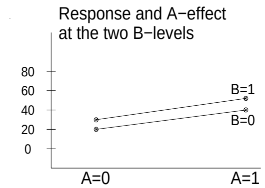
If interaction is present, then: which design is best ?

Usage of measurements: which design is best ?

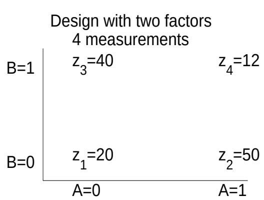
In general: How should a factorial experiment be carried out ?

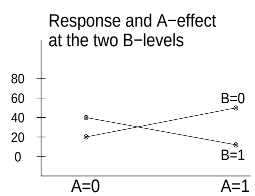
Factorial designs and interaction





The change in the response when factor A is changed is the same at both B-levels $\iff$ no interaction





The change in the response when factor A is changed depends on the B-level $\iff$ interaction

The second situation is often the case in factorial experiments

Never use one-factor-at-the-time designs. There exist better alternatives in all situations.

Blocking in factorials: Two alternative factorial designs

Complete randomization, 19th and 20th October

Additive	Temperature			
	10^oC	20^oC	30^oC	40^oC
5%	y y	y y	y y	y y
10%	y y	y y	y y	y y

$$Y_{ijk} = \mu + a_i + c_j + ac_{ij} + E_{ijk}$$

A completely randomized 2×4 factorial with two measurements per factor combination conducted over, say, two days. The design is one block of size 16.

Replication 1, October 19th

Additive	Temperature			
	10°C	20°C	30°C	40°C
5%	y	y	y	y
10%	y	y	y	y

Replication 2, October 20th

Additive	Temperature			
	10°C	20°C	30°C	40°C
5%	y	y	y	y
10%	y	y	y	y

$$Y_{ijk} = \mu + a_i + c_j + ac_{ij} + Day_k + Z_{ijk}$$

A completely randomized 2×4 factorial with one measurement per factor combination, but replicated twice, one replication per day, i.e. two blocks of size 8.

Never use the first design. Why ?

Example from Montgomery p 164

Material type	Temperature		
	15°F	70°F	125°F
1	130 155	34 40	20 70
Data	74(?) 180	80 75	82 58
Averages	$\overline{y}$=134.75	$\overline{y}$=57.25	$\overline{y}$=57.50
2	150 188	136 122	25 70
Data	159 126	106 115	58 45
Averages	$\overline{y}$=155.75	$\overline{y}$=119.75	$\overline{y}$=49.50
3	138 110	174 120	96 104
Data	168 160	150 139	82 60
Averages	$\overline{y}$=144.00	$\overline{y}$=145.75	$\overline{y}$=85.50

$$Y_{ijk} = \mu + m_i + t_j + mt_{ij} + E_{ijk}$$

A better design

Round I

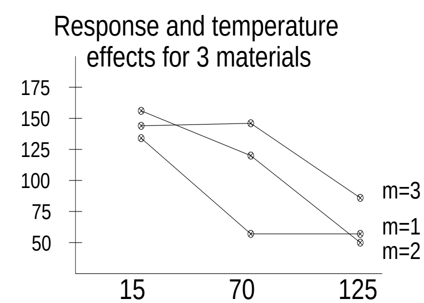
Material type 1	Temperature		
	15°F	70°F	125°F
1	y y	y y	y y
2	y y	y y	y y
3	y y	y y	y y

Round II

Material type	Temperature		
	15°F	70°F	125°F
1	y y	y y	y y
2	y y	y y	y y
3	y y	y y	y y

$$Y_{ijk} = \mu + m_i + t_j + mt_{ij} + R_k + Z_{ijk}$$

Give (at least) three reasons why this design is to be preferred.



The figure indicates a possible interaction between materials and temperature.

It is a common case that different 'materials' react differently to fx temperature treatments.

ANOVA and estimation in factorial design

$$Y_{ijk} = \mu + m_i + t_j + mt_{ij} + E_{ijk}$$

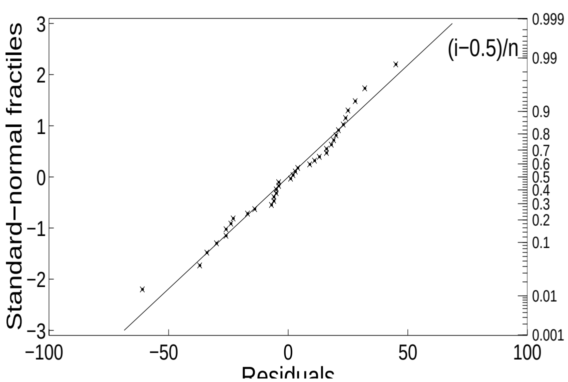
ANOVA for battery data					
Source of var.	SSQ	d.f.	s^2	EMS	F-test
Materiel	10684	2	5342	$\sigma^2 + 12\phi_m$	7.91
Temperature	39119	2	19559	$\sigma^2 + 12\phi_t$	28.97
Interaction	9614	4	2403	$\sigma^2 + 4\phi_{mt}$	3.56
Residual	18231	27	675.2	σ^2	
Total	77647	35			

$F(4,27)_{0.05} = 2.73 \Longrightarrow$ all parameters in the model are significant at the 5% level of significance.

Estimates of parameters for full model
$\widehat{\mu} = \overline{Y}_{...}$
$\widehat{m}_i = \overline{Y}_{i..} - \overline{Y}_{...}$
$\hat{t}_j = \overline{Y}_{.j.} - \overline{Y}_{...}$
$\widehat{mt}_{ij} = \overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...}$
$\widehat{\sigma}^2_E = s^2_{resid}$

Estimates of parameters for additive model
$\widehat{\mu} = \overline{Y}_{...}$
$\widehat{m}_i = \overline{Y}_{i..} - \overline{Y}_{...}$
$\hat{t}_j = \overline{Y}_{.j.} - \overline{Y}_{...}$
$\widehat{mt}_{ij} = 0$ (not in model)
$\widehat{\sigma}^2_E = (SSQ_{resid} + SSQ_{mt})/(f_{resid} + f_{mt})$

Model control based on residuals



Factorial experiments with two-level factors

The simplest example: 2 factors at 2 levels.

1. factor is called A (can be a temperature fx) (the supposedly most important factor)

2. factor is called B (can be a concentration of an additive)(the supposedly next most important factor)

For each factor combination r measurements are carried out (completely randomized):

2^2 factorial design

	B=0	B=1
A=0	Y_{001}	Y_{011}
	:	:
	Y_{00r}	Y_{01r}
A=1	Y_{101}	Y_{111}
	:	:
	Y_{10r}	Y_{11r}

$$Y_{ijk} = \mu + A_i + B_j + AB_{ij} + E_{ijk}$$

Both indices i and j can take the values '0' or '1'.

μ, A_i, B_j and AB_{ij} are the parameters of the model

Restrictions on parameters: fx $A_0 + A_1 = 0 \Longrightarrow$

$A_0 =$ - A_1 and $B_0 =$ - B_1 and

$AB_{00} =$ - $AB_{10} =$ - $AB_{01} = AB_{11}$

All parameters have only one numerical value, positive or negative, depending on the factor level(s).

Effects. Special concept for 2 level factors

Effect = change in response when the factor is changed from level '0' to '1', thus

A-effect: $A = A_1$ - $A_0 = 2A_1$ (main effect)

B-effect: $B = B_1$ - $B_0 = 2B_1$ (main effect)

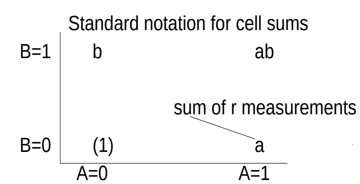
AB-effect: $AB = AB_{11}$ - $AB_{10} = 2AB_{11}$ (interaction)

In General:

k factors at 2 levels: A 2^k factorial experiment

Special notation for 2^k design

Two factors, k = 2



	B=0	B=1
A=0	(1)	b
A=1	a	ab

Fx 'a' $= \Sigma_{k=1}^{r} Y_{10k}$, the sum in the cell where the factor A is at level '1' while factor B is at level '0'.

Parameters, effects and estimation

$$\begin{array}{lll} \text{A-parameter:} & \widehat{A}_1 & = (-(1) + a - b + ab)/4r \\ \text{B-parameter:} & \widehat{B}_1 & = (-(1) - a + b + ab)/4r \\ \text{AB-parameter:} & \widehat{AB}_{11} & = (+(1) - a - b + ab)/4r \end{array}$$

$$\begin{array}{lll} \text{A-effect:} & \widehat{A} & = (-(1) + a - b + ab)/2r = 2\widehat{A}_1 \\ \text{B-effect:} & \widehat{B} & = (-(1) - a + b + ab)/2r = 2\widehat{B}_1 \\ \text{AB-effect:} & \widehat{AB} & = (+(1) - a - b + ab)/2r = 2\widehat{AB}_{11} \end{array}$$

A (very) small numerical example

Y = response = purity in solution after 48 hours

A = 1. factor = temperature (4^oC , 20^oC)

B = 2. factor = concentration of additive (5%, 10%)

	B=0	B=1
A=0	12.1	19.8
	14.3	21.0
A=1	17.9	24.3
	19.1	23.4

(1)=26.4	b=40.8
a=37.0	ab=47.7

Standard ANOVA table for example

Source of var.	SSQ	d.f.	s^2	F-value
A: temp	38.28	1	38.28	35.75
B: conc	78.75	1	78.75	73.60
AB: interaction	1.71	1	1.71	1.60
Residual	4.27	4	1.07	
Total	123.01	7		

Critical F-value: $F(1,4)_{0.05} = 7.71 \Longrightarrow$

main effects (highly) significant

interaction not significant

Estimation in detail

(1)	a	b	ab
26.4	37.0	40.8	47.7

$$\widehat{\mu} = (+26.4 + 37.0 + 40.8 + 47.7)/(2^2 \cdot 2) = 18.99$$

$$\begin{array}{lll} \widehat{A}_1 & = (-26.4 + 37.0 - 40.8 + 47.7)/(2^2 \cdot 2) & = 2.19 \\ \widehat{A}_0 & = -A_1 & = -2.19 \\ \widehat{A} & = \widehat{A}_1 - \widehat{A}_0 = 2\widehat{A}_1 & = 4.38 \end{array}$$

$$\begin{array}{lll} \widehat{B}_1 & = (-26.4 - 37.0 + 40.8 + 47.7)/(2^2 \cdot 2) & = 3.14 \\ \widehat{B}_0 & = -B_1 & = -3.14 \\ \widehat{B} & = \widehat{B}_1 - \widehat{B}_0 = 2\widehat{B}_1 & = 6.28 \end{array}$$

$$\widehat{\sigma}^2 = (SSQ_{AB} + SSQ_{resid})/(1+4) = 1.196 \simeq 1.1^2$$

(pooled estimate)

Yates algorithm, testing and estimation

Yates algorithm for k = 2 factors

Cell sums	I	II = contrasts	SSQ	Effects
(1) = 26.4	63.4	151.9 = [I]	–	$\widehat{\mu} = 18.99$
a = 37.0	88.5	17.5 = [A]	38.25	$\widehat{A} = 4.38$
b = 40.8	10.6	25.1 = [B]	78.75	$\widehat{B} = 6.28$
ab = 47.7	6.9	- 3.7 = [AB]	1.71	$\widehat{AB} = -0.93$

The important concept about Yates' algorithm is that is represents the transformation of the data to the contrasts - and subsequently to the estimates and the sums of squares!

Explanation:

Cell sums: Organized in 'standard order': (1), a, b, ab

Column I:

63.4	=	+26.4+37.0	(sum of two first in previous column)
88.5	=	+40.8+47.7	(sum of two next)
10.6	=	- 26.4+37.0	(reverse difference of two first)
6.9	=	- 40.8+47.7	(reverse difference of two next)

Column II: Same procedure as for column I (63.4+88.5=151.9)

SSQ$_A$: $[A]^2/(2^k \cdot 2) = 38.25$ (k=2) and likewise for B and AB

A-Effect: $\widehat{A} = [A]/(2^{k-1} \cdot 2) = 4.38$ and likewise for B and AB

The procedure for column I is repeated k times for the 2^k design

The sums of squares and effects appear in the 'standard order'

Numerical example with three factors (coded data)

C=0:

	B=0	B=1
A=0	- 3, - 1	- 1, 0
A=1	0, 1	2, 3

C=1:

	B=0	B=1
A=0	- 1, 0	1, 1
A=1	2, 1	6, 5

(1) = - 4	b = - 1
a = +1	ab = +5

c = - 1	bc = +2
ac = +3	abc = 11

Yates algorithm for k = 3 factors

Cell sums	I	II	III = contrasts	SSQ	Effects
(1) = - 4	- 3	1	16 = [I]	-	$\widehat{\mu} = 1.00$
a = 1	4	15	24 = [A]	36.00	$\widehat{A} = 3.00$
b = - 1	2	11	18 = [B]	20.25	$\widehat{B} = 2.25$
ab = 5	13	13	6 = [AB]	2.25	$\widehat{AB} = 0.75$
c = - 1	5	7	14 = [C]	12.25	$\widehat{C} = 1.75$
ac = 3	6	11	2 = [AC]	0.25	$\widehat{AC} = 0.25$
bc = 2	4	1	4 = [BC]	1.00	$\widehat{BC} = 0.50$
abc = 11	9	5	4 = [ABC]	1.00	$\widehat{ABC} = 0.50$

$SSQ_{resid} = [((-3)^2 + (-1)^2) - (-3-1)^2/2] + ...$

$= 2.00 + \ldots + 0.50 = 5.00$, $s^2_{resid} = SSQ_{resid}/8 = 0.625$

(variation within cells, r - 1 = 2 - 1 degrees of freedom per cell)

$SSQ_A = [A]^2/(r \cdot 2^k)$, Effect $\widehat{A} = [A]/(r \cdot 2^{k-1})$, parameter $\widehat{A}_1 = [A]/(r \cdot 2^k)$, $\widehat{A}_0 = -\widehat{A}_1$, and $\widehat{A} = 2\widehat{A}_1$.

Block designs, principles and construction

Example: factors A, B and C:

Recipes	$A_0 \Leftrightarrow$ temp $= 20^oC$	$A_1 \Leftrightarrow$ temp $= 28^oC$
	$B_0 \Leftrightarrow$ conc $= 1\%$	$B_1 \Leftrightarrow$ conc $= 2\%$
	$C_0 \Leftrightarrow$ time $= 1$ hour	$C_1 \Leftrightarrow$ time $= 2$ hours

The treatments are (1) a b ab c ac bc abc

A randomized (with respect to days) plan bc a b c (1) ab abc ac

Discussion af the randomized plan

Problem

The total time needed to carry out the plan is 1 hour for C_0 treatments and 2 hours for C_1 treatments: 2 + 1 + 1 + 2 + 1 + 1 + 2 + 2 = 12 hours.

Suggestion

Distribute the 8 experiments randomly over two days with 6 hours per day:

Day 1 $\sim$ 6 hours	Day 2 $\sim$ 6 hours
bc a b c	(1) ab abc ac

Is it balanced with respect to factors and days ?

Is this a good design ? What can go wrong ? What kind of variable is 'Days' ?

An experiment with no influence from days

Day 1: $D_1 = 0$	bc = 21	a = 23	b = 16	c = 20
Day 2: $D_2 = 0$	(1)=14	ab = 25	abc = 25	ac = 29

Cell sums	I	II	III = contrasts	SSQ	Effects
(1) = 14	37	78	173 = [I]	-	$\widehat{\mu} = 21.625$
a = 23	41	95	31 = [A]	120.125	$\widehat{A} = 7.75$
b = 16	49	18	1 = [B]	0.125	$\widehat{B} = 0.25$
ab = 25	46	13	-5 = [AB]	3.125	$\widehat{AB} = -1.25$
c = 20	9	4	17 = [C]	36.125	$\widehat{C} = 4.25$
ac = 29	9	-3	-5 = [AC]	3.125	$\widehat{AC} = -1.25$
bc = 21	9	0	-7 = [BC]	6.125	$\widehat{BC} = -1.75$
abc = 25	4	-5	-5 = [ABC]	3.125	$\widehat{ABC} = -1.25$

D_1 and D_2 are contributions from the two days (none here).

What happens if D_1 and D_2 are in fact not identical (there is a day-today effect) ?

The same experiment if D_1 and D_2 (days) in fact are different

Day 1: $D_1 = +8$	bc = 29	a = 31	b = 24	c = 28
Day 2: $D_2 = +2$	(1) = 16	ab = 27	abc = 27	ac = 31

Cell sums	I	II	III = contrasts	SSQ	Effects	Day effect
(1) = 16	47	98	213 = [I]	-	$\widehat{\mu} = 26.625$	yes
a = 31	51	115	19 = [A]	45.125	$\widehat{A} = 4.75$	yes
b = 24	59	18	1 = [B]	0.0125	$\widehat{B} = 0.25$	
ab = 27	56	1	-17 = [AB]	36.125	$\widehat{AB} = -4.25$	yes
c = 28	15	4	17 = [C]	36.125	$\widehat{C} = 4.25$	
ac = 31	3	-3	-17 = [AC]	36.125	$\widehat{AC} = -4.25$	yes
bc = 29	3	-12	-7 = [BC]	6.125	$\widehat{BC} = -1.75$	
abc = 27	-2	-5	7 = [ABC]	6.125	$\widehat{ABC} = 1.75$	yes

The experimentor cannot know (or estimate) the difference between days. The difference between days contaminates the results.

How can we place the 8 measurements on the two days in such a way that the influence from days is under control ?

Answer: Let 'Days' (blocks) follow one of the effects in the model:

$$Y_{ijk} = \mu + A_i + B_j + AB_{ij} + C_k + AC_{ik} + BC_{jk} + ABC_{ijk} + Error + \text{Day}_\ell$$

Which term could be used ? Not a main effect, but some higher order term, for example ABC (why ABC ?):

We want the confounding Blocks = ABC.

We say : defining relation I = ABC . . . but how do we do it?

Look at how contrasts for effects are calculated :

Yates algorithm - schematically - once again:

		(1)	a	b	ab	c	ac	bc	abc
$[I]$	=	+1	+1	+1	+1	+1	+1	+1	+1
$[A]$	=	- 1	+1	- 1	+1	- 1	+1	- 1	+1
$[B]$	=	- 1	- 1	+1	+1	- 1	- 1	+1	+1
$[AB]$	=	+1	- 1	- 1	+1	+1	- 1	- 1	+1
$[C]$	=	- 1	- 1	- 1	- 1	+1	+1	+1	+1
$[AC]$	=	+1	- 1	+1	- 1	- 1	+1	- 1	+1
$[BC]$	=	+1	+1	- 1	- 1	- 1	- 1	+1	+1
$[ABC]$	=	- 1	+1	+1	- 1	+1	- 1	- 1	+1

Note that any two rows are 'orthogonal' (product sum = zero).

Thus $[A]$ and $[B]$, for example, are orthogonal contrasts.

The 'index' for ABC_{ijk} is $i \cdot j \cdot k$ if indices are - 1 or + 1 like in Yates' algoritm.

Choose $\ell = i \cdot j \cdot k$ =+1 for a b c abc og -1 for (1) ab ac bc => the two blocks wanted.

The confounded block design: Blocks = ABC

Ideal data without influence from blocks:

Day 1: $D_1 = 0$	ab = 16	bc = 21	(1) = 12	ac = 20
Day 2: $D_2 = 0$	b = 24	a = 28	abc = 34	c = 22

Cell sums	I	II	III = contrasts
(1) = 12	40	80	177 = [I]
a = 28	40	97	19 = [A]
b = 24	42	8	13 = [B]
ab = 16	55	11	- 9 = [AB]
c = 22	16	0	17 = [C]
ac = 20	- 8	13	3 = [AC]
bc = 21	- 2	- 24	13 = [BC]
abc = 34	13	15	39 = [ABC]

What happens if the two days in fact influence the results differently (there is a day-to-day effect) ?

Real data with a certain influence (unknown in practice) from blocks (days):

Day 1: $D_1 = +8$	ab = 24	bc = 29	(1) = 20	ac = 28
Day 2: $D_2 = +2$	b = 26	a = 30	abc = 36	c = 24

Cell sums	I	II	III = contrasts	Day effect
(1) = 20	50	100	217 = [I]	yes
a = 30	50	117	19 = [A]	
b = 26	52	8	13 = [B]	
ab = 24	65	11	-9 = [AB]	
c = 24	10	0	17 = [C]	
ac = 28	-2	13	3 = [AC]	
bc = 29	4	-12	13 = [BC]	
abc = 36	7	3	15 = [ABC]	yes

What has changed and what has not changed? Why?

The effect from days is controlled (not eliminated) only to influence the ABC interaction term (block confounding).

Construction using the tabular method :

Arrange data in standard order and use column multiplication :

	Factor levels			Block no. =
Code	A	B	C	ABC = A·B·C
(1)	-1	-1	-1	-1
a	+1	-1	-1	+1
b	-1	+1	-1	+1
ab	+1	+1	-1	-1
c	-1	-1	+1	+1
ac	+1	-1	+1	-1
bc	-1	+1	+1	-1
abc	+1	+1	+1	+1

Block no. -1 => one block, Block no. $+1$ => the other block

Analysis of variance for block confounded design

In the example we imagine that r=2 measurements per factor combination were used. The residual SSQ is computed as the variation between these two measurements giving a total residual sum of squares with 8 degrees of freedom.

Correspondingly the responses on slide 8.8 (bottom) are sums of 2 measurements.

ANOVA for block confounded three factor design

Effects	SSQ	d.f.	s^2	F-value
A	22.56	1	22.56	9.12
B	10.56	1	10.56	4.27
AB	5.06	1	5.06	2.05
C	18.05	1	18.05	7.30
AC	0.56	1	0.56	0.22
BC	10.56	1	10.56	4.26
ABC = Blocks	14.06	1	14.06	not relevant
Residual	19.80	8	2.475	
Total	101.24	15		

$F(1,8)_{0.05} = 5.32 \Longrightarrow$ A and C main effects are significant. The B effect is only significant at the 10% level of significance, and so is BC.

The ABC effect cannot be tested because it is confounded with blocks (days) (does it seem to be a real problem ?).

A few generalizations

A 2^4 factorial design in 4 blocks of 4 :

	I_2 = BCD ~ Days			
I_1 = ABC ~	(1)	bc	d	bcd
	abd	acd	ab	ac
Operators	a	abc	b	c
	bd	cd	ad	abcd

The principal block: (1) bc abd acd

b×(1) bc abd acd = b b^2c ab^2d abcd ⇒ b c ad abcd = another block!

Multiply any block with an 'element' that is not in the block, and you get another block.

Total block variation = ABC + BCD + ABC·BCD = ABC + BCD + AD

When analyzing the data from the above 2^4 design all effects A, B, AB, ... , ABCD except ABC, BCD and AD can be estimated and tested.

ABC, BCD and AD are confounded with blocks

Construction principle : Introduce blocks into factorial by confounding

Effect		Confound
Level		
A		
B		
AB		
AC		
BC		
ABC	=	I_1
D		
AD	<=	ABC·BCD
BD		
ABD		
ACD		
BCD	=	I_2
ABCD		

All effects ABC, BCD and ABC·BCD = AD will be confounded with blocks.

Construction using the tabular method

	Factor levels						Four different	Principal
Code	A	B	C	D	ABC	BCD	blocks	block
(1)	−1	−1	−1	−1	−1	−1	1 : (−1 , −1)	(1)
a	+1	−1	−1	−1	+1	−1	2 : (+1 , −1)	
b	−1	+1	−1	−1	+1	+1	4 : (+1 , +1)	
ab	+1	+1	−1	−1	−1	+1	3 : (−1 , +1)	
c	−1	−1	+1	−1	+1	+1	4 : (+1 , +1)	
ac	+1	−1	+1	−1	−1	+1	3 : (−1 , +1)	
bc	−1	+1	+1	−1	−1	−1	1 : (−1 , −1)	bc
abc	+1	+1	+1	−1	+1	−1	2 : (+1 , −1)	
d	−1	−1	−1	+1	−1	+1	3 : (−1 , +1)	
ad	+1	−1	−1	+1	+1	+1	4 : (+1 , +1)	
bd	−1	+1	−1	+1	+1	−1	2 : (+1 , −1)	
abd	+1	+1	−1	+1	−1	−1	1 : (−1 , −1)	abd
cd	−1	−1	+1	+1	+1	−1	2 : (+1 , −1)	
acd	+1	−1	+1	+1	−1	−1	1 : (−1 , −1)	acd
bcd	−1	+1	+1	+1	−1	+1	3 : (−1 , +1)	
abcd	+1	+1	+1	+1	+1	+1	4 : (+1 , +1)	

Partially confounded 2^k factorial experiment

	B = 0	B = 1
A = 0	(1)	b
A = 1	a	ab

Suppose batches=blocks, block size = 2 :

Experiment 1 : $(1)_1 \; ab_1$ (batch 1) $a_1 \; b_1$ (batch 2) $\quad I = AB$

Model as usual: $Y_{ij\nu} = \mu + A_i + B_j + AB_{ij} + E_{ij\nu} +$ batches (blocks)

AB interaction confounded with blocks in experiment 1.

Resolving block confoundings for AB with one more experiment:

Suppose we also want to assess the interaction term AB.
We need an experiment in which AB is not confounded:

Experiment 2 : $(1)_2 \; a_2$ (batch 3) $b_2 \; ab_2$ (batch 4) $\quad I = B$

using two new batches.

Model again: $Y_{ij\nu} = \mu + A_i + B_j + AB_{ij} + E_{ij\nu} +$ batches (blocks)

The B main effect is confounded with blocks in experiment 2, but AB is not.
AB can then be estimated in experiment 2.

The price paid is that the main effect B can only be estimated in experiment 1 and AB only in experiment 2: Partial confounding.

Analyze both experiments using contrasts :

Unconfounded contrasts

$[A]_1 = -(1)_1 + a_1 - b_1 + ab_1$ (from experiment 1)

$[A]_2 = -(1)_2 + a_2 - b_2 + ab_2$ (from experiment 2)

$[B]_1 = -(1)_1 - a_1 + b_1 + ab_1$ (from experiment 1)

$[AB]_2 = +(1)_2 - a_2 - b_2 + ab_2$ (from experiment 2)

Confounded contrasts

$[AB]_1 = +(1)_1 - a_1 - b_1 + ab_1$ (from experiment 1)

$[B]_2 = -(1)_2 - a_2 + b_2 + ab_2$ (from experiment 2)

Blocks

Variation calculated as usual: From block totals

Use of unconfounded contrasts for effects:

The two (unconfounded) A-contrasts can be combined into an estimate of A and a part which expresses uncertainty:

$$[A]_{\text{total}} = [A]_1 + [A]_2 \quad \text{(both experiments combined)}$$

$$[A]_{\text{difference}} = [A]_1 - [A]_2 \quad \text{(both experiments combined)}$$

Sums of squares for A and between unconfounded A's:

$$SSQ_A = [A]^2_{\text{total}}/(2 \cdot 2^2), \quad \text{df} = 1$$

$$SSQ_{Uncert,A} = ([A]_1^2 + [A]_2^2)/2^2 - SSQ_A, \quad \text{df} = 2 - 1$$

Block (batch) totals $= T_1, T_2, T_3, T_4$, and $T_{tot} = T_1 + T_2 + T_3 + T_4$

$SSQ_{blocks} = (T_1^2 + T_2^2 + T_3^2 + T_4^2)/2 - T_{tot}^2/8$, df $= 4 - 1 = 3$.

Estimates of effects:

$\widehat{A} = [A]_{\text{total}}/(2 \cdot 2^{2-1})$ Full precision

$\widehat{B} = [B]_1/(1 \cdot 2^{2-1})$ Half precision

$\widehat{AB} = [A]_2/(1 \cdot 2^{2-1})$ Half precision

In general
$Estimate = [Contrast]/(R \cdot 2^{k-1})$
$R =$ number of times the effect is unconfounded in the experiment

Here: $R_A = 2$, $R_B = 1$, $R_{AB} = 1$, and $r = 1$ (is assumed here).

Generalization:

A 2^k factorial partially confounded, in principle as above:

Fx: $R_A =$ number of <u>unconfounded</u> A-contrasts : $[A]_1, [A]_2, \ldots, [A]_{R_A}$

Assume r repetitions (most often $r = 1$) for each response within the blocks.

$[A] = [A]_1 + [A]_2 + \ldots + [A]_{R_A}$ $\quad \widehat{A} = [A]/(R_A \cdot r \cdot 2^{k-1})$

$\widehat{A}_1 = -\widehat{A}_0 = [A]/(R_A \cdot r \cdot 2^k)$ $\quad \text{Var}\{\widehat{A}\} = \sigma^2/(R_A \cdot r \cdot 2^{k-2})$

$SSQ_A = [A]^2/(R_A \cdot r \cdot 2^k)$ $\quad f_A = 1$

$SSQ_{Uncertainty,A} = \Sigma_{i=1}^{R_A}[A]_i^2/(r \cdot 2^k) - SSQ_A$ $\quad f_{Uncertainty,A} = R_A - 1$

This calculation is done for all unconfounded effect-contrasts.

Block variation is calculated as the variation between blocks disregarding the factors. It contains block effects and confounded factor effects.

An example

Exp. 1 :	$(1)_1$=15 ab_1=7 (batch 1)	a_1=9 b_1=5 (batch 2)	$I = AB$
Exp. 2 :	$(1)_2$=11 a_2=7 (batch 3)	b_2=12 ab_2=8 (batch 4)	$I = B$
Exp. 3 :	$(1)_3$=9 b_3=11 (batch 5)	a_3=8 ab_3=6 (batch 6)	$I = A$

Model for experiment: $Y_{ij\nu} = \mu + A_i + B_j + AB_{ij} + E_{ij\nu}$+Block effects

Calculations for a twice determined effect

$[A]_1$ = - 15+7+9 - 5 = - 4 , $[A]_2$ = - 11+7 - 12+8 = - 8

$[A] = [A]_1 + [A]_2$ = - 12 , ($[A]_3 \sim$ blocks)

$\widehat{A}$ = - $12/(2 \cdot 2^{2-1})$ = - 3.0 ,

SSQ_A = (- 12)$^2/(2 \cdot 2^2)$ = 18.0 ,

$SSQ_{Uncert,A} = [(-4)^2 + (- 8)^2]/2^2$ - 18 = 2.0

Likewise for B and AB.

Completing the ANOVA table

Block totals : $T_1 = 15 + 7 = 22$, $T_2 = 9 + 5 = 14$, ... ,$[T_6] = 8 + 6 = 14$

$SSQ_{blocks} = \Sigma_i T_i^2/2 - (\Sigma_i T_i)^2/12 =$
$(22^2 + 14^2 + \ldots + 14^2)/2 - (22 + 14 + \ldots + 14)^2/12 = 28.0$, $d.f. = 5$

and it contains blocks and confounded factor effects

ANOVA table for example

Source	d.f.	SSQ	s^2	F-value	p-value	Precision
A	1	18.0	18.0	2.45	0.22	2/3
B	1	18.0	18.0	2.45	0.22	2/3
AB	1	2.0	2.0	0.27	0.64	2/3
Blocks+(A,B,AB)	5	28.0	5.60	(0.76)	(0.63)	–
Error	3	22.0	7.33			
Total	11	88.0				

Fractional 2^k designs

Example: factors A, B and C (the weights of 3 items) from 1.4 again

The complete factorial design: (1) a b ab c ac bc abc

The weighing design from 1.4 was: (1) ab ac bc

Estimate, for example: $\widehat{A}$ = [- (1) +ab +ac - bc]/2

Illustration by removing columns from contrast table

Contrasts	(1)	a	b	ab	c	ac	bc	abc
$[I]$ =	+1			+1		+1	+1	
$[A]$ =	- 1			+1		+1	- 1	
$[B]$ =	- 1			+1		- 1	+1	
$([AB])$ =	+1			+1		- 1	- 1	
$[C]$ =	- 1			- 1		+1	+1	
$([AC])$ =	+1			- 1		+1	- 1	
$([BC])$ =	+1			- 1		- 1	+1	
$([ABC])$ =	- 1			- 1		- 1	- 1	

Note: $[A] = -[BC]$, $[B] = -[AC]$, $[C] = -[AB]$ and $[I] = -[ABC]$ $\Longleftrightarrow$ Confounding of factor effects.

An alternative method of construction

Example: the complete 2^2 factorial

Effects	
I	Level
A	A-effect
B	B-effect
AB	AB-interaction

Method: Introduce the extra factor, C, by confounding it with a A, B or AB. Which one of these can (most probably) be 0 ?

Answer: AB (if any at all)

Confound C = AB $\Longleftrightarrow$ Defining relation I = ABC

A and B form the complete underlying factorial. The factor C is introduced into the complete underlying factorial as shown below:

Construction of a $1/2 \times 2^3$ design: Tabular method

There are two possibilities.

Construction : C = - AB				
2^2 design	A	B	C = - AB	$(1/2)2^3$
(1)	-	-	-	(1)
a	+	-	+	ac
b	-	+	+	bc
ab	+	+	-	ab

Construction : C = +AB				
2^2 design	A	B	C = +AB	$(1/2)2^3$
(1)	-	-	+	c
a	+	-	-	a
b	-	+	-	b
ab	+	+	+	abc

The two designs are called complementary

Together they form the complete 2^3 factorial

In general : C = $\pm$ AB can be used, i.e. two possibilities

Construction of a 2^{4-1} design

The complete underlying factorial is formed by A, B and C - the three first (most important) factors. Introduce D (the fourth factor):

Principle
$1/2 \times 2^4$
I
A
B
AB
C
AC
BC
ABC = $\pm$ D

Introduce factor D $\Rightarrow 1/2 \times 2^4$ design: Tabular method

Choose one of the possibilities, fx

2^3 codes	A	B	C	D=+ABC	2^{4-1} codes
(1)	-	-	-	-	(1)
a	+	-	-	+	ad
b	-	+	-	+	bd
ab	+	+	-	-	ab
c	-	-	+	+	cd
ac	+	-	+	-	ac
bc	-	+	+	-	bc
abc	+	+	+	+	abcd

The 2^{4-1} design contains the data code '(1)', and it is called

The principal fraction

Alias relations = Factor confoundings

Generator relation: D = +ABC $\Longrightarrow$ I = +ABCD : The defining relation

Alias relations		
I	=	+ABCD
A	=	+BCD
B	=	+ACD
AB	=	+CD
C	=	+ABD
AC	=	+BD
BC	=	+AD
ABC	=	+D

The design is a resolution IV design

Main effects and three-factor interactions confounded (often OK)

Two-factor interactions are confounded with other two-factor interactions

Analysis of data and the underlying factorial

The analysis can based on the underlying factorial (A,B,C) (forget all about D while you do the computations) :

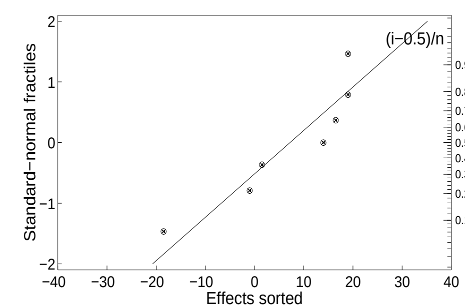
Yates algorithm for a 2^{4-1} design

Measure-ment	Response	1	2	3	Contrast	SSQ
(1) .	45	145	255	566	I + ABCD	-
a d	100	110	311	76	A + BCD	722.0
b d	45	135	75	6	B + ACD	4.5
ab .	65	176	1	-4	AB + CD	2.0
c d	75	55	-35	56	C + ABD	392.0
ac .	60	20	41	-74	AC + BD	684.5
bc .	80	-15	-35	76	BC + AD	722.0
abc d	96	16	31	66	ABC + D	544.5

The data are from page 288 (6.ed) (5.ed: 308)

Analysis of effects based on normal probability plot

The 7 estimated effects are 76/4, 6/4, ... , 66/4, respectively



The present plot does not indicate any particularly small or large effect estimates

The plot is shown to illustrate the method. With only 7 points it is difficult to conclude anything

The textbook has many more realistic examples

Example continued

Suppose it is concluded that B=0 $\Longrightarrow$ AB=0, BC=0, BD=0

It could be concluded that also CD=0 (SSQ_{AB+CD} is small)

From the beginning it was assumed that BCD=0, ACD=0, ABD=0 and ABC=0 (3 factor interactions) and ABCD=0

Remove terms corresponding to all these assumptions and conclusions from the analysis:

Reduced model analysis of variance computations

Yates algorithm for a 2^{4-1} design - final model

Measure-ment	Response	1	2	3	Effect	SSQ	Esti-mate
(1) .	45	145	255	566	μ	-	70.75
a d	100	110	311	76	A	722.0	9.50
b d	45	135	75	6		4.5	-
ab .	65	176	1	-4		2.0	-
c d	75	55	-35	56	C	392.0	7.00
ac .	60	20	41	-74	AC	684.5	-9.25
bc .	80	-15	-35	76	AD	722.0	9.50
abc d	96	16	31	66	D	544.5	8.25

Residual variance estimate

$$\widehat{\sigma}^2 = (4.5 + 2.0)/(1 + 1) = 3.25 \simeq 1.80^2$$

5 factors in 8 measurements

Construction		
I		
A		
B		
AB		
C		
AC	=	D
BC	=	E
ABC		

$\Longrightarrow$
I_1=ACD
I_2=BCE
I_1I_2=ABDE

Alias relations - all

I	=	ACD	=	BCE	=	ABDE
A	=	CD	=	ABCE	=	BDE
B	=	ABCD	=	CE	=	ADE
AB	=	BCD	=	ACE	=	DE
C	=	AD	=	BE	=	ABCDE
AC	=	D	=	ABE	=	BCDE
BC	=	ABD	=	E	=	ACDE
ABC	=	BD	=	AE	=	CDE

In defining relation **I = ACD = BCE = ABDE** at least 3 letters in all terms (except I) $\Rightarrow$ Resolution is III.

Alias relations for model without high order interactions

Without many-factor interactions

I						
A	=	CD				
B			=	CE		
AB					=	DE
C	=	AD	=	BE		
AC	=	D				
BC			=	E		
(ABC)	=	BD	=	AE		

Construction of $1/4 \times 2^5$ design

Underlying Factorial				Introduce			× de ⇒
				D	E	Design	Principal
Code	A	B	C	=+AC	=+BC	2^{5-2}	fraction
(1)	-	-	-	+	+	de	(1)
a	+	-	-	-	+	ae	ad
b	-	+	-	+	-	bd	be
ab	+	+	-	-	-	ab	abde
c	-	-	+	-	-	c	cde
ac	+	-	+	+	-	acd	ace
bc	-	+	+	-	+	bce	bcd
abc	+	+	+	+	+	abcde	abc

In the principal fraction D = - AC and E= - BC, and t here are 4 possible (equally usefull) designs:

$$\begin{aligned} D &= \pm AC \\ &\text{combined with} \\ E &= \pm BC \end{aligned}$$

A 2^{5-2} factorial in 2 blocks of 4

Use fx D= - AC and E= - BC and Blocks=ABC :

Alias relations reduced						
I	=	- ACD	=	- BCE		+ABDE
A	=	- CD				
B			=	- CE		
AB					=	+DE
C	=	- AD	=	- BE		
AC	=	- D				
BC			=	- E		
(ABC)	=	- BD	=	- AE	=	Blocks

Construction using the tabular method

2^3	A	B	C	D= - AC	E= - BC	2^{5-2}	ABC=Block
(1)	-	-	-	-	-	(1)	- ∼ 1
a	+	-	-	+	-	ad	+ ∼ 2
b	-	+	-	-	+	be	+ ∼ 2
ab	+	+	-	+	+	abde	- ∼ 1
c	-	-	+	+	+	cde	+ ∼ 2
ac	+	-	+	-	+	ace	- ∼ 1
bc	-	+	+	+	-	bcd	- ∼ 1
abc	+	+	+	-	-	abc	+ ∼ 2

Design:

Block 1	Block 2
(1) abde ace bcd	ad be cde abc
Even ABC	Uneven ABC

Example 8-6 p 308, design construction

```
Construction of design by introducing the factors F, G and H into the complete factorial defined
by A, B, C, D and E. The design is carried out in 4 blocks.

I
A
B
AB
C
AC
BC
ABC   = +F
D
AD
BD
ABD   = +G
CD
ACD
BCD           = Blocks
ABCD
E
AE
BE
ABE           = Blocks
CE
ACE
BCE
ABCE
DE
```

```
ADE
BDE
ABDE
CDE
ACDE         = Blocks  ( BCD*ABE = ACDE )
BCDE  = +H
ABCDE
```

Example 8-6 p 308, statistical analyses

```
Analysis of 2**k complete and fractional factorial designs

This program was prepared by Henrik Spliid
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Lyngby, DK-2800, Denmark. (hs@imm.dtu.dk)
Version: 25/08/99

file=Montg_ex8-6.dat, Edited September 1, 2001. Course F-343, DFH.
Input for this problem was read from file  Montg_ex9-6.dat
Output was written to file                 Montg_ex9-6.out

The 10 factors A - J are treated in the design
The  5 factors A - E define the complete underlying factorial structure
The  3 factors F - H are embedded in the underlying factorial structure
The  2 factors I - J define the blocking
```



```
Description and options given by user :
Confoundings:(F=ABC,G=ABD,H=BCDE,Blocks=ABE=ACDE),
Options:(LaTeX Dispersion)

The treatments of the experiment were :
  h              afghj          bfgij          abi
  cfi            acgij          bcghj          abcfh
  dgi            adfij          bdfhj          abdgh
  cdfgh          acdhj          bcdij          abcdfgi
  ej             aefg           befghi         abehij
  cefhij         aceghi         bceg           abcefj
  deghij         adefhi         bdef           abdegj
  cdefgj         acde           bcdehi         abcdefghij

Data ordering in relation to standard order is :
   1              2              3              4
   5              6              7              8
   9             10             11             12
  13             14             15             16
  17             18             19             20
  21             22             23             24
  25             26             27             28
  29             30             31             32

From the treatments given above the following confoundings have been computed.
Interactions between factors and blocks assumed = zero
Max. factorial interaction order considered =  3
```



```
Alias relations to interaction order 3 :
*
A     = +BCF   = +BDG
B     = +ACF   = +ADG
AB    = +CF    = +DG
C     = +ABF   = +DFG
AC    = +BF    = +EGH
BC    = +AF    = +DEH
ABC   = +F     = +CDG
D     = +ABG   = +CFG
AD    = +BG    = +EFH
BD    = +AG    = +CEH
ABD   = +CDF   = +G
CD    = +FG    = +BEH
ACD   = +BDF   = +BCG   = +AFG
BCD   = +ADF   = +ACG   = +BFG   = +EH    = Blocks
ABCD  = +DF    = +CG    = +AEH
E
AE    = +DFH   = +CGH
BE    = +CDH   = +FGH
ABE   = +CEF   = +DEG   = Blocks
CE    = +BDH   = +AGH
ACE   = +BEF   = +GH
BCE   = +AEF   = +DH
ABCE  = +EF    = +ADH   = +BGH
DE    = +BCH   = +AFH
ADE   = +BEG   = +FH
BDE   = +AEG   = +CH
ABDE  = +EG    = +ACH   = +BFH
CDE   = +EFG   = +BH
ACDE  = +ABH   = +CFH   = +DGH   = Blocks
BCDE  = +H
ABCDE = +DEF   = +CEG   = +AH
Note: Design has resolution IV
==============================
```



```
From the treatments given above the following confoundings have been computed.
Interactions between factors and blocks assumed = zero
Max. factorial interaction order considered =  2
Alias relations to interaction order 2 :
*
A
B
AB    = +CF    = +DG
C     =
AC    = +BF
BC    = +AF
ABC   = +F
D     =
AD    = +BG
BD    = +AG
ABD            = +G
CD    = +FG
ACD
BCD                                  = +EH    = Blocks
ABCD  = +DF    = +CG
E
AE
BE
ABE                     = Blocks
CE
ACE            = +GH
BCE            = +DH
ABCE  = +EF
DE
ADE            = +FH
BDE            = +CH
ABDE  = +EG
CDE            = +BH
ACDE                    = Blocks
BCDE  = +H
ABCDE                   = +AH
```

10.6

Response: Log-SD

Printout of input data:

			Response	Code	A	B	C	D	E	F	G	H	I	J
1	(1)	=	1.02	h	0	0	0	0	0	0	0	1	0	0
2	a	=	1.82	afghj	1	0	0	0	0	1	1	1	0	1
3	b	=	.89	bfgij	0	1	0	0	0	1	1	0	1	1
4	ab	=	1.39	abi	1	1	0	0	0	0	0	0	1	0
5	c	=	.91	cfi	0	0	1	0	0	1	0	0	1	0
6	ac	=	1.78	acgij	1	0	1	0	0	0	1	0	1	1
7	bc	=	.87	bcghj	0	1	1	0	0	0	1	1	0	1
8	abc	=	1.21	abcfh	1	1	1	0	0	1	0	1	0	0
9	d	=	1.48	dgi	0	0	0	1	0	0	1	0	1	0
10	ad	=	1.41	adfij	1	0	0	1	0	1	0	0	1	1
11	bd	=	1.17	bdfhj	0	1	0	1	0	1	0	1	0	1
12	abd	=	1.33	abdgh	1	1	0	1	0	0	1	1	0	0
13	cd	=	1.67	cdfgh	0	0	1	1	0	1	1	1	0	0
14	acd	=	1.35	acdhj	1	0	1	1	0	0	0	1	0	1
15	bcd	=	1.11	bcdij	0	1	1	1	0	0	0	0	1	1
16	abcd	=	1.08	abcdfgi	1	1	1	1	0	1	1	0	1	0
17	e	=	.97	ej	0	0	0	0	1	0	0	0	0	1
18	ae	=	1.70	aefg	1	0	0	0	1	1	1	0	0	0
19	be	=	.81	befghi	0	1	0	0	1	1	1	1	1	0
20	abe	=	1.45	abehij	1	1	0	0	1	0	0	1	1	1
21	ce	=	.94	cefhij	0	0	1	0	1	1	0	1	1	1
22	ace	=	1.68	aceghi	1	0	1	0	1	0	1	1	1	0
23	bce	=	.75	bceg	0	1	1	0	1	0	1	0	0	0
24	abce	=	1.43	abcefj	1	1	1	0	1	1	0	0	0	1
25	de	=	1.38	deghij	0	0	0	1	1	0	1	1	1	1
26	ade	=	1.18	adefhi	1	0	0	1	1	1	0	1	1	0
27	bde	=	1.23	bdef	0	1	0	1	1	1	0	0	0	0
28	abde	=	1.46	abdegj	1	1	0	1	1	0	1	0	0	1
29	cde	=	1.49	cdefgj	0	0	1	1	1	1	1	0	0	1
30	acde	=	1.29	acde	1	0	1	1	1	0	0	0	0	0
31	bcde	=	1.48	bcdehi	0	1	1	1	1	0	0	1	1	0
32	abcde	=	1.22	abcdefghij	1	1	1	1	1	1	1	1	1	1

10.7

Effects and aliases		no.	Sum of Squares	Deg.fr.	Effect estimates
*		0	52.4032	1	1.2797
A		1	.6641	1	.2881
B		2	.3180	1	-.1994
A B		3	.0003	1	-.0056
C		4	.0058	1	-.0269
A C		5	.0294	1	-.0606
B C		6	.0167	1	-.0456
A B C	= + F	7	.0124	1	-.0394
D		8	.0914	1	.1069
A D		9	1.1213	1	-.3744
B D		10	.0226	1	.0531
A B D	= + G	11	.1093	1	.1169
C D		12	.0088	1	.0331
A C D		13	.0248	1	-.0556
B C D	= + I	14	.0102	1	-.0356
A B C D		15	.0017	1	-.0144
E		16	.0000	1	-.0019
A E		17	.0004	1	.0069
B E		18	.0790	1	.0994
A B E	= + J	19	.0088	1	.0331
C E		20	.0124	1	.0394
A C E		21	.0003	1	.0056
B C E		22	.0020	1	.0156
A B C E		23	.0026	1	-.0181
D E		24	.0026	1	.0181
A D E		25	.0063	1	-.0281
B D E		26	.0282	1	.0594
A B D E		27	.0215	1	-.0519
C D E		28	.0011	1	.0119
A C D E		29	.0011	1	-.0119
B C D E	= + H	30	.0014	1	.0131
A B C D E		31	.0205	1	-.0506
Total for Effects			2.6247	31	

10.8

```
             Normal probability plot for estimates
    HS +I--------I---------I---------I---------I---------I+
 3.0000-                                                  -
       I                                                  I
       I                                                  I
       I                                                  I
       I                                        (A)  X    I
 2.0000-                                                  -
       I                                                  I
       I                                  X               I
       I                                 X                I
       I                                 X                I
 1.0000-                              2                   -
       I                             X                    I
       I                           X2                     I
       I                           2                      I
       I                          XX                      I
  .0000-                          3                       -
       I                         2                        I
       I                        XX                        I
       I                       X2                         I
       I                       X                          I
-1.0000-                       2                          -
       I                      X                           I
       I                      X                           I
       I        (B)  X                                    I
       I                                                  I
-2.0000-                                                  -
       I X  (AD)                                          I
       I                                                  I
       I                                                  I
       I                                                  I
-3.0000-                                                  -
    HS +I--------I---------I---------I---------I---------I+
       -.3900             -.1050               .1950
                -.2550               .0450              .3450
```


10.9

Estimates for final model:

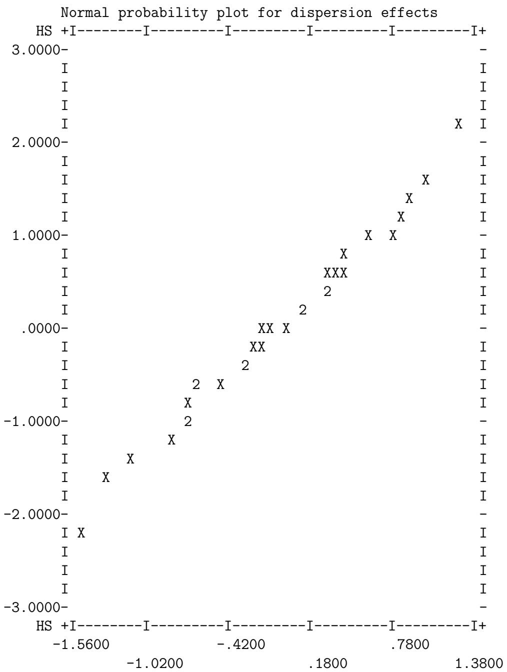
Effect		Estimate	Stand. dev.	t-test
Grand mean		1.2797	.0202	>99.95%
A		.2881	.0404	>99.95%
B		-.1994	.0404	>99.95%
D		.1069	.0404	98.5 %
A D	(= BG ?)	-.3744	.0404	>99.95%
A B D	= + G	.1169	.0404	99.1 %
B C D	= + I (blocks)	-.0356	.0404	61.3 %
A B E	= + J (blocks)	.0331	.0404	57.9 %
A C D E	(blocks)	-.0119	.0404	22.8 %

Variability	Estimate	Stand. dev.
Residual standard dev.	.1143	.0169
Coefficient of variation	8.93 %	1.32 %

For computing the residual variance and standard deviation the sums of squares for the significant effects and the blocks are subtracted from the total variation giving the residual sum of squares with 32-1-8 = 23 degrees of freedom.

A model control can be made using the dispersion effects See Montgomery p. 239 and 300.

The plot below does not indicate any dispersion effects :



Summary of analyses

Significant effects: A, B, D, AD, G (could be the conclusion)

Term	Levels	Effect	Parameters
μ	constant	1.28	1.28
A	0 - 15 × 0.001 inch	0.29	[- 0.145, +0.145]
B	0 - 15 × 0.001 inch	- 0.20	[+0.10, - 0.10]
D	tool vendor	0.11	[- 0.055, +0.055]
AD	interaction	- 0.37	[+0.185, - 0.185]
G	0 - 15 × 0.001 inch	0.12	[- 0.06, +0.06]
σ^2	residual variance	0.11^2	0.11^2

Combining main effect and interaction effect estimates

The estimate of the combined effect af A, D and AD is :

A_0= - 0.145	AD_{00}= - 0.185	AD_{01}=+0.185
A_1=+0.145	AD_{10}=+0.185	AD_{11}= - 0.185
	D_0= - 0.055	D_1=+0.055

giving (- 0.145 - 0.055 - 0.185 = - 0.385, for example) :

A = 0	- 0.385	+0.095
A = 1	+0.275	+0.015
	D = 0	D = 1

The minimum response is wanted (it is log-standard deviation). Choose A=0, D=0, B=1, G=0 (the combination with lowest estimated response)

Model identified

$$Y_{ijkl} = \mu + A_i + B_j + D_k + AD_{ik} + G_l + \epsilon_{ijkl}$$