Exercise in Factor Analysis

Consider the <u>independent</u> random variables

$$\mathbf{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \in \mathcal{N}(\mathbf{0}, \mathbf{I}) = \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$\mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} \in \mathcal{N}\left(\mathbf{0}, \mathrm{diag}(\frac{4}{9}, \frac{7}{9}, \frac{4}{9}) \right) = \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{4}{9} & 0 & 0 \\ 0 & \frac{7}{9} & 0 \\ 0 & 0 & \frac{4}{9} \end{pmatrix} \right)$$

and define $\mathbf{X}$ as

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$$

or shortly

$$\mathbf{X} = \mathbf{AF} + \mathbf{G}$$

This is the factor model with observation $\mathbf{X}$, factor loadings $\mathbf{A}$, common factors $\mathbf{F}$, and unique factors $\mathbf{G}$, cf. textbook p. 274–275.

1. What are the 3 communalities?
2. What is the distribution of $\mathbf{X}$?
3. What are the squared multiple correlations $\rho^2_{1|23}$, $\rho^2_{2|13}$ and $\rho^2_{3|12}$ for the $\mathbf{X}$ coordinates?
4. Show that
$$[D(\mathbf{X})]^{-1} = \frac{1}{71} \begin{pmatrix} 80 & -27 & 3 \\ -27 & 81 & -9 \\ 3 & -9 & 72 \end{pmatrix}$$
5. Determine the distribution of
$$\begin{pmatrix} \mathbf{F} \\ \mathbf{X} \end{pmatrix} = (F_1, F_2, X_1, X_2, X_3)'$$
6. Determine the conditional distribution of
$$(F|\mathbf{X})$$
7. How may we predict $\mathbf{F}$ if we have observed $\mathbf{X}$?
8. What are the uncertainties on the predicted values?