Week 8

02407 Stochastic Processes
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Renewal processes are embedded in regenerative processes. An example of a renewal process is the successive returns to a recurrent state in a Markov chain (discrete time) or Markov jump process (continuous time).

If we consider a point process on the line, where the distances between points are described as independent exponential random variables all with the same rate/intensity then we have the interval characterisation of the Poisson process. To get a renewal-process we relax the assumption of exponentiality. We let $X_n, 1 \leq n$ denote the independent random variables separating the points while W_n is the time of the occurrence of the nth point. The time of the nth occurrence W_n is given by $W_n = \sum_{i=1}^n X_i$. The renewal process $N(t) = \max\{n \geq 0 : W_n \leq t\}$ with $W_0 = 0$ counts the number of points occurring up to and including time t. The distribution of $N(t)$ is surprisingly hard to calculate in the non-Poissonian case, however, assuming phase-type or matrix-exponentially distributed time intervals leads to a matrix-based tractable numerical expression for $\Pr\{N(t) = n\}$. For general distributions usually only expressions for the moments of $N(t)$ are derived, in particular $E[N(t)] = M(t)$. Even these expressions can only be found as (e.g Laplace) transforms, via the formula

$$\int_0^\infty e^{-\theta t} \mathrm{d}M(t) = \frac{L(\theta)}{\theta(1 - L(\theta))}$$

where $L(\theta) = E\left[e^{-\theta X_n}\right] = \int_0^\infty e^{-\theta t} \mathrm{d}F(t)$. The renewal function has a linear asymptote and a sufficiently scaled version has the normal distribution as limiting distribution.

Another frequently studied object is the Renewal equation $A(t) = a(t) + \int_0^t A(t-u)\mathrm{d}F(u)$. It has the solution $A(t) = \int_0^t a(t-u)\mathrm{d}M(u) \to \frac{1}{\mu}\int_0^\infty a(t)\mathrm{d}t$, where $A(t) = M(t)$ for $a(t) = 1$.

The concepts of length biased sampling and moment distributions give important insight in sampling and relations between random variables and derived distributions. These concepts are closely related with the distributions of residual life time, age, and total life time (spread) in a renewal process.

One reason to study phase type distributions is that these provide examples of stochastic processes amenable for numerical computation. In class I will introduce most concepts of Chapter 7 using phase type distributions deviating somewhat from the presentation of the text book.

Must read and nice to read

The processes of residual life time, total life time (or spread), and age defined in Section 7.1 are important. The renewal function $M(t)$ of Equation (7.6) is generally not well suited for numerical evaluation, and the expression has primarily theoretical interest. As Section 7.2 contains examples, the section is more motivational than essential. Section 7.3 can be viewed as an additional example section. The Poisson process is the only renewal process with scalar computable formulae for all quantities of interest. In my view, Section 7.4 is the most important section of the chapter. The section contains the Elementary Renewal Theorem, Equation (7.13), and results on the asymptotic distribution of a properly normalised version of $N(t)$ along with the previously mentioned important limiting results for residual life time, age, and total life time. The case of a modified, also called delayed, renewal process is treated in Section 7.5. In this case the time X_1 to the first occurrence is allowed to follow a general distribution. Of particular importance is the equilibrium or stationary renewal process where the distribution of X_1 is the same as the limiting distribution of the residual life time. The renewal process with discrete index set is treated in Section 7.6. As the discrete case is frequently more accessible than the continuous from a theoretical point of view, Section 7.6 can aid in understanding the concepts of renewal theory. However, I will not cover this Section in class.

Some important definitions and results

Renewal process: Waiting time to nth event (7.1) p.347

$W_n = X_1 + X_2 + \ldots + X_n, \quad F(x) = \Pr\{X_k \le x\}, \quad X_i$ independent

Renewal process: Counting process (7.2) p.347

$\{N(t); t \ge 0\}, \quad N(t) = \max\{n \ge 0 | W_n \le t\}$

Renewal function (7.6) p.349

$$M(t) = E[N(t)] = \sum_{k=1}^{\infty} \Pr\{W_k \le t\} = \sum_{k=1}^{\infty} F_k(t), \qquad F_k(t) = \int_0^t F_{k-1}(t-x)\mathrm{d}F(t)$$

Mean arrival time of next event (7.7) p.349

$E[W_{N(t)+1}] = E(X_1)(E(N_t)+1) = \mu(M(t)+1)$

Elementary renewal theorem (7.13) p.363

$$\lim_{t\to\infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

Elementary renewal theorem random variable version

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ almost surely}$$

Renewal density for continuous $F(x)$ (7.15) p.366

$$m(t) = \frac{\mathrm{d}M(t)}{\mathrm{d}t} = \sum_{k=1}^{\infty} f_k(t), \quad f_k(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t}$$

Refinement of elementary renewal theorem for finite $E(X^2)$ (7.17) p.367

$$\lim_{t\to\infty} \left(M(t) - \frac{t}{\mu}\right) = \frac{\sigma^2 - \mu^2}{2\mu^2}, \quad Var(X_k) = \sigma^2$$

Asymptotic distribution p.368

$$\lim_{t\to\infty} \Pr\left\{\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \le x\right\} = \Phi(x)$$

Definition of excess, age, and total life p.349

$\gamma_t = W_{N(t)+1} - t, \quad \delta_t = t - W_{N(t)}, \quad \beta_t = \gamma_t + \delta_t = W_{N(t)+1} - W_{N(t)}$

Limiting distribution of residual life time (7.21) p.368

$$\lim_{t\to\infty} \Pr\{\gamma_t \le x\} = \frac{1}{\mu}\int_0^x (1 - F(u))\mathrm{d}u$$

Limiting distribution of joint distribution of age and residual life time p.369

$$\lim_{t\to\infty} \Pr\{\gamma_t \ge x, \delta_t \ge y\} = \frac{1}{\mu}\int_{x+y}^{\infty} (1 - F(z))\mathrm{d}z$$

Limiting distribution of total life time (spread)

$$\lim_{t\to\infty} \Pr\{\beta_t \le x\} = \frac{\int_0^x t\mathrm{d}F(t)}{\mu}$$

Delayed renewal process, distribution of X_1 can be different

Stationary renewal processes (7.26) p.372

$$M_D(t) = E(N(t)) = \frac{t}{\mu}, \quad \Pr\{\gamma_t \le x\} = \frac{1}{\mu}\int_0^x (1 - F(X))\mathrm{d}x = G(x)$$

Cumulative and related processes (7.28) p.374

$$W(t) = \sum_{k=1}^{N(t)} Y_k, \quad \lim_{t\to\infty} \frac{E(W(t))}{t} = \frac{E(Y_1)}{\mu}$$