If we consider a point process on the line, where the distances between points are described as independent exponential random variables all with the same rate/intensity then we have the interval characterisation of the Poisson process.

To get a renewal process we relax the assumption of exponentiality.

We let \( X_n, 1 \leq n \) denote the independent random variables separating the points while \( W_n \) is the time of the occurrence of the \( n \)th point.

\[
W_n = \sum_{i=1}^{n} X_i.
\]

Elementary renewal theorem

\[
\frac{M(t)}{t} \to \frac{1}{\mu}
\]

\[
E(W_{N(t)+1}) = \mu (1 + M(t))
\]

\[
M(t) = \sum_{n=1}^{\infty} F_n(t), \quad F_n(t) = \int_{0}^{t} F_{n-1}(t-x) dF(t)
\]

Renewal equation

\[
A(t) = a(t) + \int_{0}^{t} A(t-u) dF(u)
\]

Solution to renewal equation

\[
A(t) = \int_{0}^{t} a(t-u) dM(u) \to \frac{1}{\mu} \int_{0}^{\infty} a(t) dt
\]

What are the main results

Renewal function Length biased sampling - moment distributions Generalisation of Poisson process Exact distribution of counts surprisingly hard to find Ordinary/modified/delayed Next week see class of distributions where one can at least numerically do something.

Definition Exact renewal function - also LST Elementary renewal theorem \( E(W_{N(t)}) \) Asymptotic normality \( H(t) \) Asymptotic linear Limit residual life time (age) - also joint Limit spread
Some important definitions and results

Renewal process: Waiting time to $n$th event (7.1) p.347

$W_n = X_1 + X_2 + \ldots + X_n, \quad F(x) = \Pr\{X_k \leq x\}, \quad X_i$ independent

Renewal process: Counting process (7.2) p.347

$\{N(t); t \geq 0\}, \quad N(t) = \max\{n \geq 0|W_n \leq t\}$

Renewal function (7.6) p.349

$M(t) = E[N(t)] = \sum_{k=1}^{\infty} \Pr\{W_k \leq t\} = \sum_{k=1}^{\infty} F_k(t)$

Mean arrival time of next event (7.7) p.349

$E[W_{N(t)+1}] = E(X_1)(E(N_t) + 1) = \mu(M(t) + 1)$

Elementary renewal theorem (7.13) p.363

$$\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}$$

Elementary renewal theorem random variable version

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ almost surely}$$

Renewal density for continuous $F(x)$ (7.15) p.366

$$m(t) = \frac{dM(t)}{dt} = \sum_{k=1}^{\infty} f_k(t), \quad f_k(t) = \frac{dF(t)}{dt}$$

Refinement of elementary renewal theorem for finite $E(X^2)$ (7.17) p.367

$$\lim_{t \to \infty} \left( M(t) - \frac{t}{\mu} \right) = \frac{\sigma^2 - \mu^2}{2\mu^2}, \quad \text{Var}(X_k) = \sigma^2$$

Asymptotic distribution p.368

$$\lim_{t \to \infty} \Pr\left\{ \frac{N(t) - t/\mu}{\sqrt{\frac{\sigma^2}{\mu^3}}} \leq x \right\} = \Phi(x)$$

Definition of excess, age, and total life p.349

$\gamma_t = W_{N(t)+1} - t, \quad \delta_t = t - W_N, \quad \beta_t = \gamma_t + \delta_t = W_{N(t)+1} - W_N$

Limiting distribution of residual life time (7.21) p.368

$$\lim_{t \to \infty} \Pr\{\gamma_t \leq x\} = \frac{1}{\mu} \int_0^x (1 - F(u))du$$

Limiting distribution of joint distribution of age and residual life time p.369

$$\lim_{t \to \infty} \Pr\{\gamma_t \geq x, \delta_t \geq y\} = \frac{1}{\mu} \int_{x+y}^\infty (1 - F(z))dz$$

Limiting distribution of total life time (spread)

$$\lim_{t \to \infty} \Pr\{\beta_t \leq x\} = \frac{\int_0^t t dF(t)}{\mu}$$

Delayed renewal process, distribution of $X_1$ can be different

Stationary renewal processes (7.26) p.372

$M_D(t) = E(N(t)) = \frac{t}{\mu}, \quad \Pr\{\gamma_t \leq x\} = \frac{1}{\mu} \int_0^x (1 - F(X))dx = G(x)$

Cumulative and related processes (7.28) p.374

$W(t) = \sum_{k=1}^{N(t)} Y_k, \quad \lim_{t \to \infty} \frac{E(W(t))}{t} = \frac{E(Y_1)}{\mu}$