Week 7

02407 Stochastic Processes
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BFN/bfn

Phase-type distributions are the topic of this week. These are defined as the time to absorption in finite state Markov chains (discrete time) or Markov jump processes (continuous time).
One can alternatively think of these distribution as compositions of of geometric or exponential life-times. In this form the distributions can be traced back to Ellis while they are usually ascribed to Erlang. In the modern - Markov chain/jump process - formulation they are usually ascribed to Neuts who invented the term, while the idea goes back to Jensen, so there is a strong Danish contributing to the development.
One of my favourite quotes by Neuts is "Most research is about clarification" and this is certainly true for phase-type distributions. The first results are immediate implications of the general theory, while results on e.g. closure properties are harder to find in the general theory on Markov processes. Our study of the topic can broadly be classified as follows Definition, Basic properties including distribution form, transforms and moments, Closure properties, Phase type renewal processes (which we will treat next week as an introduction to general renewal processes).

Definition

For discrete variables: $Y = \min\{n : X_n = r\}$. For continuous variables: $Y = \min\{n : X(t) = r\}$. A phase type distribution is parameterised through a representation $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{S})$. Keep in mind that a representation is generally not unique which is a severe complication for statistical inference, but sometimes an advantage in stochastic modelling.

Basic properties

For discrete variables: $\mathbb{P}(Y > n) = \boldsymbol{\alpha}\boldsymbol{S}^n\mathbf{1}$, $\mathbb{E}\left(\theta^Y\right) = \alpha_{p+1} + \theta\boldsymbol{\alpha}\left(\boldsymbol{I} - \theta\boldsymbol{S}\right)^{-1}\boldsymbol{s}$, $\mathbb{E}(Y) = \mu = \boldsymbol{\alpha}\left(\boldsymbol{I} - \boldsymbol{S}\right)^{-1}\mathbf{1}$.
For continuous variables: $\mathbb{P}(Y > t) = \boldsymbol{\alpha}e^{\boldsymbol{S}t}\mathbf{1}$, $\mathbb{E}\left(e^{-\theta Y}\right) = \alpha_{p+1} + \boldsymbol{\alpha}\left(\theta\boldsymbol{I} - \boldsymbol{S}\right)^{-1}\boldsymbol{s}$, $\mathbb{E}(Y) = \mu = \boldsymbol{\alpha}\left(-\boldsymbol{S}\right)^{-1}\mathbf{1}$

Closure properties

The class of phase type distributions are closed under a number of operations. Expressed in terms of the random variables it is sums, formation of order statistics including minimum and maximum, random sums with a (discrete) phase type distributed number of terms, and randomly selected variables. Formulated in terms of the distributions it is convolutions, (discrete) phase type weighted countable convolutions and mixtures. Also moment distributions from phase type distributions are of phase type. Most results can be proven by probabilistic arguments. However, currently a probabilistic argument only exists for the first order moment distribution. Also, the derivation of the representations for the moment distributions are more involved than for the other properties, at least with current technique of proof. The results for moments distributions are somewhat recent and due to Mogens Bladt and me.

Phase type renewal processes

A renewal process with phase type distributed inter event times induces a Markov jump process (continuous time Markov chain) by the concatenation of the paths of the successive processes. The infinitesimal generator is given by

$$\boldsymbol{A} = \boldsymbol{S} + \boldsymbol{s}\boldsymbol{\alpha}$$

The limiting residual life time distribution, i.e. the distribution of δ_∞ is of phase type: $\delta_\infty \sim \mathrm{PH}(\boldsymbol{\pi}, \boldsymbol{S})$ where $\boldsymbol{\pi}$ is the invariant or limiting - if you like - probability distribution of $X(t)$ obtained by solving $\boldsymbol{\pi}\boldsymbol{A} = \mathbf{0}$. The invariant probability vector $\boldsymbol{\pi}$ has an explicit solution $\boldsymbol{\pi} = \boldsymbol{\alpha}\left(-\boldsymbol{S}\right)^{-1}/\mu$