Week 6

02407 Stochastic Processes
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BFN/bfn

The general class of continuous time Markov chains (Markov jump processes) are treated in Section 6.6 albeit restricted to the case of finite state space. The theory generalises but is more demanding for a fully rigorous treatment in the countable state space case.

Three different characterisations are discussed. The first characterisation is through the semi-group property expressed via the Chapman-Kolmogorov equation in (c) Page 327. The second characterisation is the infinitesimal characterisation given at the bottom of Page 328. Finally, the process can be understood as a randomisation of the time between state transitions in a discrete time Markov chain, where the parameter of the exponential distribution depends on the current state of the (embedded) Markov chain. This third characterisation is described at the top of Page 329.

The first result of primary importance is Equation (6.68) $\mathbf{0} = \boldsymbol{\pi A}$ or element-wise rephrased into $\pi_j q_j = \sum_{i \neq j} \pi_i q_{ij}$ that expresses that the "the probabilistic flow" out of a state is equal to the flow into the state "in equilibrium". A slightly more technical rephrasing is that the expected number of jumps out of $j$ per time unit is equal to the expected number of jumps into $j$ under the invariant or stationary distribution. Equation (6.68) can be referred to as the global balance equations. For some models, e.g. Birth and Death Processes, the more restrictive local balance equations $\pi_i q_{ij} = \pi_j q_{ji}$ hold. The Markov jump process is reversible when the local balance equations hold. The second important result is $\boldsymbol{P}(t) = \exp(\boldsymbol{A}t)$ expressed in Equation (6.67).

A main application of Markov jump processes is queueing theory. The basic concepts of queueing theory like the queue length and waiting time are introduced in Section 9.1 and related in this section through The Queueing Formula (Little's law) $L = \lambda W$. A number of classical yet important queueing models M/M/1, M/M/∞, and M/M/s are described in Section 9.2. These queueing models in Section 9.2 with countable state space can all be formulated as Birth and Death Processes. The invariant queue length distribution of the M/M/1 is geometric, while the invariant queue length distribution of the M/M/∞ queue is Poisson. The finite variants with blocked customers have truncated geometric and truncated Poisson distributions.

Poisson Arrivals See Time Averages (the PASTA property) as explained in the Wikipedia page on The Arrival Theorem is important and quite useful but not explicitly covered by Pinsky and Karlin.

The M/G/1 queue is treated in Section 9.3. Neither the queueing process nor the workload process are Markov processes. However, the processes embedded at time epochs immediately after departures are Embedded Markov processes. The first moment of these processes are expressed in the Pollaczek-Khinchine formula (9.35) and (9.36).

**Must read and nice to read**

All material in Section 6.6 up to the Example *Industrial Mobility and the Peter Principle* should be studied. The rest of the section contains two examples that are nice to know but not essential. In Chapter 9 all of sections 9.1-9.3 is relevant material. I will probably not have time to cover the $M/G/\infty$ model in class.

**Some important definitions and results**

Finite state Markov jump process (continuous time Markov chain) (6.58) p.327

$$P_{ij}(t) = \Pr\{X(t+s) = j | X(s) = i\}, \quad P_{ik}(s+t) = \sum_{j=0}^{N} P_{ij}(s)P_{jk}(t), \quad \sum_{j=0}^{N} P_{ij}(t) = 1, \quad \lim_{t \to 0+} P_{ij}(t) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Infinitesimal description bottom of p.328

$$\Pr\{X(t+h) = j | X(t) = i\} = q_{ij}h + o(h), \quad \Pr\{X(t+h) = i | X(t) = i\} = 1 - q_i h + o(h), \text{ with } q_i = \sum_{j \neq i} q_{ij}$$

Exponential sojourn times and embedded Markov chain description top of p.329

Matrix differential equations (6.66) and (6.67) p.329

$$\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{A} = \boldsymbol{A}\boldsymbol{P}(t), \quad \boldsymbol{P}(t) = e^{\boldsymbol{A}t}, \text{ with } e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n t^n}{n!}$$

Global balance equations (6.68) p.330

$$\mathbf{0} = \boldsymbol{\pi}\boldsymbol{A}, \quad \pi_j q_j = \sum_{i \neq j} \pi_i q_{ij}, j = 0, 1, \ldots, N$$

The queueing equation (Little's law) 9.1.1 p.448

$L = \lambda W, \quad L_0 = \lambda W_0$ with $W = W_0 +$ mean service time

Shorthand (Kendall) notation p.449 $A/B/c$

Queue length distribution for $M/M/1$ queue (9.11) and (9.12) p.453

$$\pi_k = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k, \quad L = \mathrm{E}(X(t)) = \frac{\lambda}{\mu - \lambda}, \text{ assuming time invariant solution}$$

Waiting time distribution for $M/M/1$ queue bottom of page 454 and (9.15) p.455

$$\Pr\{T \leq t\} = 1 - e^{-t(\mu - \lambda)}, \quad W = \mathrm{E}(T) = \frac{1}{\mu - \lambda}$$

Queue length distribution for $M/M/\infty$ queue (9.16) p.456

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} e^{-\frac{\lambda}{\mu}}, \quad L = \mathrm{E}(X(t)) = \frac{\lambda}{\mu}, \text{ assuming time invariant solution}$$

Performance measures for $M/M/s$ queue (9.19) p.458

$$L_0 = \frac{\pi_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\frac{\lambda}{s\mu}}{\left(1 - \frac{\lambda}{s\mu}\right)^2} \quad W_0 = \frac{L_0}{\lambda}, \quad W = W_0 + \frac{1}{\mu}, \quad L = \lambda W = L_0 + \frac{\lambda}{\mu}$$

Embedded Markov chain for $M/G/1$ queue (9.24) p.462

$$X_n = \begin{cases} X_{n-1} - 1 + A_n & \text{if } X_{n-1} > 0 \\ A_n & \text{if } X_{n-1} = 0 \end{cases} = (X_{n-1} - 1)^+ + A_n, \qquad X_n \text{ queue length after } n\text{th departure}$$

Performance measures for $M/G/1$ queue (Pollaczek–Khinchine formula) (9.35) p.464 and (9.36) p.465

$$L = \rho + \frac{\lambda^2(\tau^2 + v^2)}{2(1 - \rho)}, \quad W = v + \frac{\lambda(\tau^2 + v^2)}{2(1 - \rho)}, \quad v = \mathrm{E}(Y_n), \quad \tau^2 = \mathrm{Var}(Y_n), \quad Y_n \text{ is a generic service time}, \quad \rho = \lambda v$$