Week 5

02407 Stochastic Processes
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We studied discretely indexed Markov processes with discrete state space in Chapter 4 (Markov Chains). These processes can be analysed with a relatively moderate mathematical level. Discretely indexed Markov processes on a general state space requires somewhat more advanced techniques, see [4]. These processes are also termed Markov chains, where the properties of the state space is usually understood from the context. The theory is underlying the statistical discipline Markov Chain Monte Carlo (MCMC), although only a limited number of professionals working with MCMC will have studied the theoretical development of such processes.

In Chapter 6 we will be exposed to continuously indexed Markov processes on a discrete state space. These processes are frequently referred to as Continuous time Markov Chains (CTMC), however personally I find the term Markov Jump processes better suited [1, 2]. The one-dimensional Poisson process of Chapter 5 is the basic example of such processes. An in-depth treatment of Markov jump processes (Continuous time Markov chains) requires surprisingly advanced mathematical techniques. This requirement stems from the needs of a proper analysis of the equation

$$\mathbb{P}\{X(t) = j | \{X(u)\}_{0 \leq u \leq s}\} = \mathbb{P}\{X(t) = j | X(s)\}.$$

An ultimate, but rarely studied, source is [3]. Alternatively one can start from the embedded Markov chain of jumps and an assumption of exponential sojourn times in each state. The traditional way of performing this analysis is to make some further assumptions to obtain what is called a standard process.

Pinsky and Karlin has chosen to present infinite (countable) state space models only for the special case of birth-and-death processes while the general continuous time Markov chains are assumed to be on finite state spaces in their presentation.

The starting point in Section 6.1 on the pure birth process is the characterisation of the Poisson process in terms of infinitesimal probabilities/intensity postulates. These intensities are made dependent on the number of points that have occurred so far, while the assumption of absolute randomness is otherwise retained. Counting events by subtracting one unit (deaths) rather than adding one unit (births) one obtains the pure death process described in Section 6.2. In Section 6.3 both positive and negative contributions are allowed (births and deaths) to obtain the class of birth-and-death processes. These processes are extremely useful and provide the framework for many specific stochastic models, particularly in queueing theory, as we shall see in Chapter 9. Section 6.4 is devoted to the limiting behaviour of birth-and-death processes.

The embedded Markov chain of a birth-and-death process is irreducible more or less by construction, if not it is very easy to identify the irreducible classes. The transition probabilities in a birth-and-death process converges i.e. $\lim_{t\to\infty} P_{ij}(t) = \pi_j$ (Equation (6.30)) exist and are independent of $i$ as in the discrete (irreducible) case. If the limit is positive then the process is positive recurrent and the limiting probability distribution also has the role of the invariant or stationary probability distribution (Equations (6.31)).

**Must read and nice to read**

In Section 6.1 on the pure birth process you can read the last discussion of the Yule process on Page 283 lightly. Example 6.2.2 on Cable Failure under Static Fatigue in Section 6.2 on pure death processes is nice but not of ultimate importance for the flow of the text. It is probably only necessary to study one of the examples of Section 6.4 with care, I find the example on the *Repairman model* most straightforward. The rest of the material, I think, needs to be studied with some care.

**Some important specific definitions and results**

Postulates of birth and death process (see also (6.1) p.278, (6.12) p.287) p.295

$P_{ij}(t) = \Pr\{X(t+s) = j | X(s) = i\}, \quad P_{i,i+1}(t+h) = \lambda_i h + o(h), \quad P_{i,i-1}(t+h) = \mu_i h + o(h)$

Relation between arrival times $W_k$, sojourn times $S_i$, and number of births p.279

$$P_n(t) = \Pr\{W_n \leq t < W_{n+1}\} = \Pr\left\{\sum_{i=0}^{n-1} S_i < t \leq \sum_{i=0}^{n} S_i\right\}$$

State probabilities for pure birth process (6.4,6.5) p.279,280

$$P_0(t) = e^{-\lambda_0 t}, \quad P_n(t) = \Pr\{X(t) = n\} = \lambda_{n-1} e^{-\lambda_n t} \int_o^t e^{\lambda_n x} P_{n-1}(x) \mathrm{d}x \quad n \geq 1$$

Criterion for finiteness (avoiding explosion) (6.6) p.280

$$\sum_{n=0}^{\infty} P_n(t) = 1 \Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty$$

Exponentiality of sojourn times (see also (6.1) p.278, (6.12) p.287) p.297

$\Pr\{S_i \geq t\} = e^{-(\lambda_i + \mu_i)t}$

Criterion for well-definedness of birth and death process defined by intensities (6.21) p.297

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n \theta_n} \sum_{k=0}^{n} \theta_k = \infty, \quad \theta_0 = 1, \quad \theta_n = \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}$$

Existence of limit for irreducible birth and death process (6.31) p.304

$$\lim_{t \to \infty} P_{ij}(t) = \pi_j \geq 0$$

Equivalence of invariant and limiting distribution for irreducible birth and death process (6.32,6.33) p.305

$$\sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}(t), \forall t \geq 0$$

Invariant (and limiting) probabilities (6.37) p.306

$$\pi_0 = \left(\sum_{k=0}^{\infty} \theta_k\right)^{-1}, \quad \pi_j = \theta_k \pi_0 = \frac{\theta_j}{\sum_{k=0}^{\infty} \theta_k}, \quad \theta_j = \prod_{k=0}^{j-1} \frac{\lambda_k}{\mu_{k+1}}$$

**Some typos**

Page 305 (6.36) $\mu_J$ should be $\mu_j$

# References


[1] Søren Asmussen. *Applied Probability and Queues*. Springer–Verlag, New York, second edition, 2003.

[2] Mogens Bladt and Bo Friis Nielsen. *Matrix-Exponential Distributions in Applied Probability*. Matrix-exponential Distributions in Applied Probability. Springer, 2017.

[3] Chung Kai-Lai. *Markov Chains with Stationary Transition Probabilities*. Springer-Verlag, New York, 1967.

[4] Sean Meyn and Richard L. Tweedie. *Markov Chains and Stochastic Stability*. ambridge University Press, New York, NY, USA, 2 edition, 2009.