Week 4

02407 Stochastic Processes
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BFN/bfn

This week we study the Poisson Process. The Poisson Process is one of the fundamental building blocks in probability theory and stochastic processes. You might already be familiar with its properties, however, we will most likely study them in more depth than you have seen before.

Three different ways of defining the Poisson process on the line are introduced.

**Independent increments** In terms of independent Poisson distributed increments, definition bottom of Page 225

**Intensity/infinitesimal probabilities** In terms of infinitesimal probabilities Definition Page 234

**Sequence of independent exponential intervals** In terms of independent exponential sojourn times Theorem 5.5 Page 243.

These are the three most important definitions, they are all equivalent, which might not be completely clear from the text book.

Section 5.1.1 recalls basic results for the Poisson distribution, while Section 5.1.2, containing the definition of the Poisson process as an integer valued stationary stochastic process with independent increments, is the most important subsection of Section 5.1. Poisson processes with randomly varying intensity as described in Section 5.1.4 are important and there is a vast literature on those. Some exercises later in the course will treat a special case.

Binomial probabilities are well approximated by Poisson probabilities for $\mu = np$ with $n$ large and $\mu$ moderate (much smaller than $n(1-p)$). The law of rare events - Theorem 5.3 - of Section 5.2 generalises this concept and motivates why the Poisson process is frequently a surprisingly precise model of real world phenomena, like the central limit theorem explains the usability of the normal distribution. The law of rare events is used to motivate the definition of the Poisson (counting) process from infinitesimal probabilities. Pinsky and Karlin introduces the Poisson point process as a process of the increments, however, this is slightly subtle, and you can safely concentrate on the Poisson (counting) process $\{X(t); t \geq 0\}$ as the formulation in terms of $N((t_k, t_{k+1}])$ is only justified by slightly more elegant derivations at places.

The third definition (or characterisation) in terms of independent exponential inter event times is presented in Section 5.3. The essence of that section is contained in the three theorems 5.4-5.6.

Theorem 5.6 of Section 5.3 provides a bridge to Section 5.4. The Poisson process can be thought of - or is - the model of complete randomness which is reflected in Theorem 5.7. Knowing that we have $n$ points from a Poisson process tells us that the position of each individual point - say the $k$th point has position $U_k$ - is given by the uniform distribution. The naturally sorting of the points provided by their successive occurrences gives us the arrival times $W_k$ which is then given as the order statistics of the $U_k$s, $W_k = U_{(k)}$. Various applications are presented as examples throughout the remainder of the section. The sections 5.4.1 and 5.4.2 contain a couple of more elaborate examples.

With a minor change the definition of the Poisson process from the independent increment property carries over to more general index sets. It is more natural to do this as a Poisson point process rather than a Poisson counting process due to the lack of ordering in general index sets. The definition in terms of infinitesimal probabilities - intensities - can likewise be generalised while the third approach with exponential inter arrival times is hard to generalise although it should be possible in Euclidean spaces (I think). The more general index set is the topic of Section 5.5.

If we associate a random variable $Y_k$ with the $k$th point we obtain the compound Poisson Process as the cumulated values $Z(t) = \sum_{k=1}^{X(t)} Y_k$, with the $Y_k$s being mutually independent and independent of $X(t)$. The process of the pairs $(W_1, Y_1), (W_2, Y_2), \ldots$ is called a marked Poisson process. For fixed $t$ the compound Poisson process can be analysed as a random sum. Recall the transform methods are convenient particularly to obtain higher order moments. The two definitions are the most important material of Section 5.6.

**Must read and nice to read**

Most of the material is definitely worth reading. However, as there is a lot of material I will give some suggestions for lighter reading In Section 5.1 is 5.1.1 known material, while 5.1.3 and 5.1.4 can be read lightly. The proof of Theorem 5.2.3 in Section 5.2 is nice but is not essential as long as you understand the result. The proof, however, contains important arguments and will definitely help improve your skills in probabilistic reasoning. I recommend that you study either 5.4.1 or 5.4.2 with some care but not necessarily both sections. Section 5.5 can be read lightly. The two last examples of Section 5.6 can be read lightly and even be skipped, if you are really pressed for time.

**Some important specific definitions and results**

Definition of Poisson process p.225

$\{X(t); t \geq 0\}$ with independent increments $X(t_n) - X(t_{n-1}) \sim \text{Pois}\left(\lambda\left(t_n - t_{n-1}\right)\right), \quad X(0) = 0$

Theorem 5.3 The law of rare events (5.7) p.233

$$\left|\Pr\{S_n = k\} - \frac{\mu^k}{k!}e^{-\mu}\right| \leq \sum_{i=1}^{n} p_i^2, \quad S_n = \sum_{i=1}^{n} Z_i, \quad Z_i \in \{0,1\}, \quad \Pr(Z_i = 1) = p_i$$

Definition of Poisson (counting) process p.234

Definition of Poisson point process p.236

$N\left((t_0, t_1]\right), N\left((t_1, t_2]\right) N\left((t_{m-1}, t_m]\right)$ independent $N((s,t]) \sim \text{Pois}(\lambda(t-s))$

Theorem 5.4 Time to $n$th event is gamma distributed (5.12) p.242

$$f_{W_n}(t) = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}e^{-\lambda t}, \quad W_n \text{ is time of } n\text{th event}, \quad f_{W_1}(t) = \lambda e^{-\lambda t}$$

Theorem 5.5 Independent exponentially distributed sojourn times (5.14) p.243

$$f_{S_k}(t) = \lambda e^{-\lambda t}, \quad S_k = W_{k+1} - W_k, \quad W_0 \stackrel{\text{def}}{=} 0$$

Theorem 5.6 Conditional distribution is binomial (5.16) p.244

$$\Pr\{X(u) = k | X(t) = n\} = \binom{n}{k}\left(\frac{u}{t}\right)^k \left(\frac{t-u}{t}\right)^{n-k}, \quad 0 < u \leq t, \quad 0 \leq k \leq n$$

Theorem 5.7 Uniformity of time epochs conditioned on total number (5.18) p.248

$$f_{W_1,\ldots,W_n|X(t)=n}(w_1,\ldots,w_n) = \frac{n!}{t^n}, \quad 0 \leq w_1 \leq \cdots \leq w_n \leq t$$

Definition of Compound Poisson process (5.31) p.264

$$Z(t) = \sum_{k=1}^{X(t)} Y_k, \quad G(y) = \Pr(Y_k \leq y), \quad Y_k \quad \text{mutually independent and independent of } X(t)$$

Definition of Marked Poisson process p.267

The sequence of pairs, $\quad (W_1, Y_1), (W_2, Y_2), \ldots$