Week 3

02407 Stochastic Processes
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BFN/bfn

The highlight of Chapter 4 are Theorems 4.3 and 4.4. These results tell us that under certain regularity conditions a Markov chain will have a limiting distribution $\boldsymbol{\pi}$ that can be found as the solution to the fixed point problem $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$, the so-called time-invariant or stationary solution. An element π_i of $\boldsymbol{\pi}$ can thus be interpreted as

1. The limit $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$
2. The reciprocal of the mean return time $m_j = \pi_j^{-1}$ from state j back to itself.
3. The long run expected time $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(M_j^{(n)}|X_0 = i) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)}$ spent in state j per time unit.

The two theorems are presented in Section 4.4, the regularity conditions needed for them are in Section 4.3. The first important concepts is that state j is accessible from i if there exists k such that $P_{ij}^{(k)} > 0$, while i and j communicates if j is accessible from i and i is accessible from j. In words: State i and j communicate if it is possible to get from one of the states to the other and back again. The state space can then be partitioned in sets of communicating states (communicating classes) and a remainder set of states that do not communicate with any other states. First return and first passage probabilities are introduced in Section 4.3.3 P.198. The first return time distribution is the distribution of the time to the first return to state i while initiating in i. The first passage distribution is the distribution of the time of the first visit to a state $j \neq i$, where i is the initiating state. In chapter 3 we have studied the time to absorption T. The distribution of T is an example of a first passage distribution. A state is called recurrent if the first return time distribution is proper, that is has probability mass 1, while a state is called transient if the first return time distribution is defect, that is has probability mass less than 1. It is proved in Section 4.3 that all states in a communicating class is of the same type, thus we can speak about the type of a communicating class. A Markov chain that consists of only one communicating class of recurrent states is called irreducible. Some communicating classes and thus irreducible Markov chains are periodic, such that states can only be visited at certain times. A little bit of care is then needed as only the limits $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(M_j^{(n)}|X_0 = i) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)}$ exist.

Must read and nice to read

Section 4.1 is a warm-up or motivating section describing particularly benign Markov chains where the main results of the chapter can be proved without too much effort. Several quite illustrative examples are presented in Section 4.2. You don't have to read this Section but I recommend that you devote at least a little bit of time to understand the basics of some of the examples like Sections 4.2.1 and 4.2.4. Sections 4.3 and 4.4 are important and you should study them with great care. Unless you are very short of time you should study the examples carefully too, The two examples concluding Section 4.4 are a bit long and technical but will be worth your time (I find). Section 4.5 discusses reducible Markov chains and how the analysis of such can be boiled down to the study of its communicating classes. The properties of interest will be the probability of reaching the different communication classes and the time it takes to reach such a class. Finally, once the Markov chain has reached a communication class the behaviour of each communication class can be studied in isolation.

Some important specific definitions and results

Regular transition matrix: limiting probabilities (4.2) and (4.3) p.168

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, j = 0, \ldots, N, \quad \sum_{k=0}^{N} \pi_k = 1$$

First passage probabilities p.198

$$f_{ij}^{(n)} = \Pr\{X_n = j, X_v \neq i, v = 1, 2, \ldots, n-1 | X_0 = i\}$$

Relation between first passage and n-step transition probabilties (4.16) p.168

$$P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}, \quad n \geq 1$$

Theorem 4.2 A state i is recurrent if and only if p.199

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$$

Corollary 4.3 If $i \leftrightarrow j$ and if i is recurrent then j is recurrent p.200

Theorem 4.3 The basic limit theorem of Markov chains (4.26) p.204

$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i}, \ m_i \text{ is mean recurrence time}$$

Theorem 4.4 Invariant distribution theorem of Markov chains p.204

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad \sum_{i=0}^{\infty} \pi_i = 1$$

Theorem 4.4 Uniqueness of invariant distribution (4.27) p.204

$$\pi_i \geq 0, \quad \sum_{i=0}^{\infty} \pi_i = 1, \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$$

Some typos

Page 169 lb.1 $x = \pi_l \to x_l = \pi_l$

Page 194 The second transition matrix should have been $\mathbf{P} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$.

Page 198 l.8 In other words, $P_{ii}^{(n)} \to$ In other words, $f_{ii}^{(n)}$

Page 199 lb.4 Now, equation (3.5) $\to$ Now, equation (4.19)

Page 207 lt.5 $p_1 \to p_2$