Week 2

02407 Stochastic Processes
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BFN/bfn

In week two we focus on some important technical results. These definitions and results are of intrinsic value and additionally we will need several of them throughout the course. We will focus on the concepts of random walks, matrix formulation and interpretation of first passage probabilities, random sums, and generating functions.

Must read and nice to read

Random walks are studied in Section 3.6. We will focus on the initial setup leading to Equation (3.46), and Definition (3.49) leading to Equation (3.50). The two main results are (3.48) for the absorption probabilities and (3.52) for the expected time to absorption in the case $p = q = \frac{1}{2}$. Even though the derivations leading to these results are somewhat neat I will not spend time on these somewhat tedious derivations. You do not need to study these derivations, you can focus on understanding the model.

Section 3.7 is concerned with a direct evaluation of the expected time to absorption in an absorbing chain. The formulae for the absorption probabilities then follow. This is an alternative way of deriving these two properties (expected time and absorption probabilities) and we will tie a link between the two approaches. I find the section useful for several reasons. The use of indicator variables as applied in this section is commonly used in probability theory. This application of indicator variables is a good support for the intuition about and understanding of Markov chains and nth step transition probabilities. The highlights of the section can be viewed as Equations (3.83) and (3.84), which are matrix relations between nth step transition probabilities and expected time spent in the transient states before absorption. These equations then lead to (3.92) and the expression $\boldsymbol{U} = \boldsymbol{W}\boldsymbol{R}$ for the absorption probabilities.

Random sums, treated in Section 2.5 occur frequently. The formulae for the mean and variance of such a sum are special cases of of the formulae for conditional expectation and variance. We will spend some time discussing these. The probability generating function described in Sections 3.9.1 and 3.9.2 is an important analytical tool. We will use branching processes of Sections 3.8 and 3.9 as the motivating example. However, I would like you to focus your attention on the concept of generating functions rather than the branching processes. The latter can be studied in their own right, but they are not particularly important for this course.

Some important specific definitions and results

One-Dimensional random walk (3.38) p.116

Probability of absorption at 0 in RW with barriers at 0 and N (3.48) p.116

$$u_k = \begin{cases} \frac{N-k}{N} & \text{when } p = q = \frac{1}{2} \\ \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1-\left(\frac{q}{p}\right)^N} & \text{when } p \neq q \end{cases}$$

Expected time spent in state j in first n time steps, starting in i (3.80) p.140

$$W_{ij}^{(n)} = \mathrm{E}\left[\sum_{\ell=0}^{n} \mathbf{1}\{X_\ell = j\} | X_0 = i\right]$$

Recursive expression for $\boldsymbol{W}^{(n)}$ (3.83) p.141

$$\boldsymbol{W}^{(n)} = \boldsymbol{I} + \boldsymbol{Q}\boldsymbol{W}^{(n-1)}$$

Expected time spent in state j before absorption, starting in i (3.84) p.142

$$\boldsymbol{W} = \boldsymbol{I} + \boldsymbol{Q}\boldsymbol{W}$$

Probability of absorption in k not later than time n starting in i (3.92) p.142

$\boldsymbol{U}^{(n)} = \boldsymbol{W}^{(n-1)}\boldsymbol{R}$

Probability of ultimate absorption in k starting in i (elementwise version of $\boldsymbol{U} = \boldsymbol{W}\boldsymbol{R}$) (3.93) p.143

$U_{ik} = \sum_{j=0}^{r-1} W_{ij} R_{jk},$ for $0 \le i < r$ and $r \le k \le N$

Conditional expectation - discrete random variable (2.6) p.50

$\mathrm{E}[g(X)] = \mathrm{E}\left\{\mathrm{E}[g(X)|Y]\right\}$

Law of total variance (not immediately in Pinsky & Karlin)

$\mathrm{Var}[X] = \mathrm{E}\left\{\mathrm{Var}[X|Y]\right\} + \mathrm{Var}\left\{\mathrm{E}[X|Y]\right\}$

Definition of random sum (2.22) p.57

$X = \sum_{k=1}^{N} \xi_k$ (empty sum defined to be 0)

Mean and variance of random sum (2.30) p.59

$\mathrm{E}[X] = \mu\nu, \quad \mathrm{Var}[X] = \nu\sigma^2 + \mu^2\tau^2, \quad$ with $\quad \mathrm{E}[\xi_k] = \mu, \quad \mathrm{Var}[\xi_k] = \sigma^2, \quad \mathrm{E}[N] = \nu, \quad \mathrm{Var}[N] = \tau^2$

Definition of probability generating function (3.103) p.152

$\phi(s) = \mathrm{E}\left[s^{\xi}\right] = \sum_{k=0}^{\infty} p_k s^k,$ for $0 \le s \le 1$ with $p_k = \Pr(\xi = k)$

Probability generating function of sum of independent variables (3.105) (see also (3.108)) p.153

$\phi_X(s) = \mathrm{E}\left[s^X\right] = \prod_{i=1}^{n} \phi_i(s),$ with $X = \sum_{i=1}^{n} \xi_i, (\xi_i$ independent) and $\phi_i(s) = \mathrm{E}\left[s^{\xi_i}\right]$

Expectation from generating function (3.106) (factorial moments) p.153

$$\left.\frac{\mathrm{d}\phi(s)}{\mathrm{d}s}\right|_{s=1} = \mathrm{E}[\xi], \quad \left.\frac{\mathrm{d}^2\phi(s)}{\mathrm{d}s^2}\right|_{s=1} = \mathrm{E}[\xi(\xi-1)], \quad \left.\frac{\mathrm{d}^m\phi(s)}{\mathrm{d}s^m}\right|_{s=1} = \mathrm{E}\left[\prod_{k=0}^{m-1}(\xi - k)\right]$$

Variance from generating function (3.107) p.154

$$\mathrm{Var}[\xi] = \left.\frac{\mathrm{d}^2\phi(s)}{\mathrm{d}s^2}\right|_{s=1} - \left(\left.\frac{\mathrm{d}\phi(s)}{\mathrm{d}s}\right|_{s=1}\right)^2 + \left.\frac{\mathrm{d}\phi(s)}{\mathrm{d}s}\right|_{s=1} = \phi''(1) - \phi'(1)^2 + \phi'(1)(\text{ assuming existence of moments})$$

Probability generating function of random sum (3.110) p.157

$\phi_X(s) = \phi_N(\phi_\xi(s))$

Some typos

Page 60, line 7 from the bottom should have been

$$= \sum_{n=1}^{\infty} \sigma^2 n p_N(n) = \nu\sigma^2$$

Page 144 Exercise 3.7.1 lines 12 and 11 from bottom. States should be labelled 1 and 2 rather than 0 and 1.