Week 12

02407 Stochastic Processes
2023-11-20
BFN/bfn

Black Scholes Geometric Brownian motion is an important model in financial engineering, maybe most notably expressed in the Black-Scholes-Merton model.

Ohrnstein-Uhlenbeck process

Brownian measure and integration

Some important definitions and results

Black Scholes pricing (8.56) p.429

$$F(z;\tau) = z\Phi\left(\frac{\log\left(\frac{z}{a}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) - ae^{-r\tau}\Phi\left(\frac{\log\left(\frac{z}{a}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)$$

Ohrnstein-Uhlenbeck process (8.57) p.433

$$V(t) = ve^{-\beta t} + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta t}} B\left(e^{2\beta t} - 1\right) \text{ for } t \geq 0\}, \quad (\{B(t); t \geq 0\} \text{ standard Brownian motion})$$

Ohrnstein-Uhlenbeck process: mean and variance (8.58) and (8.59) p.433

$$E[V(t)] = ve^{-\beta t}, \quad Var[V(t)] = \sigma^2\left(\frac{1 - e^{-2\beta t}}{2\beta}\right)$$

Ohrnstein-Uhlenbeck process: Covariance function (8.61) p.433

$$Cov[V(u), V(s)] = \frac{\sigma^2}{2\beta}\left(e^{-\beta(s-u)} - e^{-\beta(s+u)}\right), 0 < u < s$$

Ohrnstein-Uhlenbeck process: infinitessimal definition (8.62) and (8.63) p.434

$$E(\Delta V | V(t) = v) = -\beta v \Delta t + o(\Delta t), \quad Var(\Delta V | V(t) = v) = \sigma^2 \Delta t + o(\Delta t)$$

Integrated Ohrnstein-Uhlenbeck process (position process) (8.64) p.437

$$S(t) = S(0) + \int_0^t V(u)\mathrm{d}u \text{ for } t \geq 0\}, \quad (\{V(t); t \geq 0\} \text{ Ohrnstein-Uhlenbeck process})$$

Integrated Ohrnstein-Uhlenbeck process: mean and variance (8.65) p.438

$$E[S(t)] = 0, \quad Var[V(t)] = \frac{\sigma^2}{\beta^2}\left[t - \frac{2}{\beta}\left(1 - e^{-\beta t}\right) + \frac{1}{2\beta}\left(1 - e^{-2\beta t}\right)\right], \text{ with } S(0) = V(0) = 0$$

Stationary Ohrnstein-Uhlenbeck process (limiting behavoir too) (8.68) p.439

$$V^s(t) = \frac{\sigma e^{-\beta t}}{\sqrt{2\beta t}} B\left(e^{2\beta t}\right) \text{ for } -\infty < t < \infty 0\}, \quad (\{B(t); t \geq 0\} \text{ standard Brownian motion})$$

Covariance function stationary Ohrnstein-Uhlenbeck process (8.69) p.439

$$Cov[V^s(s), V^s(t)] = \frac{\sigma^2}{2\beta} e^{-\beta|t-s|}$$

Some typos

Page 555 Solution 8.5.2 numbers should be multiplied by 2 Page 445 Exercise 8.5.3 (a) The question should be: Determine the mean value function and correlation function for $\{V_n\}$. (this is then in accordance with the solution on Page 555).