Week 11

02407 Stochastic Processes
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BFN/bfn

Several models derived from Brownian motion are described in Section 8.3 and Section 8.4. The two sections are rich in results. A condensed summary follows.

Reflected Brownian motion $\{R(t), 0 \leq t\}$, where $R(t) = |B(t)|$. Here $\mathbb{E}(R(t)) = \sqrt{2t/\pi}$, $\mathbb{V}\text{ar}(R(t)) = (1 - 2t/\pi)$, and the transition density is $p(y, t|x) = \phi\left(\frac{y+x}{\sqrt{t}}\right) + \phi\left(\frac{y-x}{\sqrt{t}}\right)$.

Absorbed Brownian motion $\{A(t), 0 \leq t\}$. With τ being the time of absorption, or first passage time to the origin the process $A(t)$ is defined to be equal to $B(t)$ for $t < \tau$ and 0 for $\tau < let$. By use of the reflection principle we get

$$\mathbb{P}(A(t) > y|A(0) = x) = \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right), \qquad \mathbb{P}(A(t) = 0) = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

Brownian bridge The finite dimensional distributions of the Brownian bridge $\{B^0(t), 0 \leq t \leq 1\}$ are obtained from Brownian motion by conditioning on $B(1) = 0$. As Brownian motion is a Gaussian process, we find the conditional distribution $\mathbb{P}\left(B^0(t) \leq x|B(1) = 0\right)$ to be given by the normal distribution with mean 0 and variance $t(1-t)$. The Brownian bridge is thus also a Gaussian process. The covariance is found to be $\mathbb{C}\text{ov}(B^0(s), B^0(t)) = s(1-t)$ for $0 \leq s \leq t \leq 1$. The test statistic in the *Kolmogorov-Smirnov test* for goodness of fit is derived from the Brownian bridge. This is closely related to the example given.

Brownian meander $\{B^+(t), 0 \leq t\}$ is derived from Brownian motion by conditioning on the process being positive. Using the results from $\{A(t), 0 \leq t\}$ one find e.g $\mathbb{P}\left(B^+(t) > y|B^+(0) = 0\right) = \exp\left(-y^2/(2t)\right)$.

Brownian motion with drift $\{X(t), 0 \leq t\}$ with $X(t) = \mu t + \sigma B(t)$.

Absorption probability and mean time to absorption With two barriers at a b like in Section 3.6 (there 0 and N), $u(x)$ is the probability of ultimate absorption at the upper barrier b at absorption time T starting from x. In Section 3.6 the quantity was named u_k due to the discrete state space, but conceptually there is no difference. With some simplifying assumptions a second order differential equation rather than a second order difference equation is derived, ultimately leading to Theorem 8.1. A similar approach is used to obtain Theorem 8.2 on the mean time to absorption. Finally, the result for $u(x)$ leads to the exponential formula for the maximum of an unrestricted Brownian motion with negative drift, $M = \max_{0 \leq t}\{X(t)\}$ $\mathbb{P}(M > x) = \exp\left(-2|\mu|x/\sigma^2\right)$.

Geometric Brownian motion A positive random variable Z is defined to be log-normally distributed $Z \sim \text{LN}(\kappa, \beta^2)$ if the natural logarithm of the variable $X = \log(Z) \sim \text{N}(\kappa, \beta^2)$, or if starting from X we get define $Z = \exp(X)$. The distribution of sums of random variables converge to the normal distribution according to the central limit theorem. The distribution of products of random variables converge to the log-normal distribution, under similar assumptions for the individual terms in the product. Geometric Brownian motion $Z(t) = z\exp(\mu t + \sigma B(t))$ can be seen as the stochastic process version of this relation. Most results can be transferred from the Brownian motion with drift regime by taking the logarithm of $Z(t)$, perform calculations for $X(t) = \mu t + \sigma B(t)$ and then transform back using exponentiation. An interesting oddity is that it is straightforward to construct a process where $Z(t) \to 0$ with probability one, while $\mathbb{E}(Z(t)) \to \infty$.

Must read and nice to read
Like the rest of the chapter Section 8.3 and Section 8.4 are tightly written. You can skip or read the example on the Black Scholes formula ligthly. The example is important in financial engineering,

awarding a Nobel prize, but the derivation of Formula (8.53) is omitted (for good reason) and the rest is merely tedious but relevant calculations.

Some additional important definitions and results

Reflected Brownian motion p.411

$\{R(t); t \geq 0\}, \quad R(t) = |B(t)| \quad (\{B(t); t \geq 0\}$ standard Brownian motion

Reflected Brownian motion: mean and variance (8.27) and (8.28) p.412

$$E[R(t)] = \sqrt{\frac{2t}{\pi}}, \quad Var[R(t)] = \left(1 - \frac{2}{\pi}\right) t$$

Reflected Brownian motion: transition kernel p.412

$$p(y,t|x) = \phi\left(\frac{y-x}{\sqrt{t}}\right) + \phi\left(\frac{y+x}{\sqrt{t}}\right)$$

Absorbed Brownian motion p.412

$A(t) = B(t)1\{t \leq \tau\}, \quad \tau = \min\{t \geq 0 | B(t) = 0\}$

Distribution of absorbed Brownian motion (8.32) p.414

$$\Pr\{A(t) > y | A(0) = x\} = \Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right), \quad \Pr\{A(t) = 0 | A(0) = x\} = 2\left(1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right)$$

Brownian meander p.416

$$\Pr\{B^+(t) > y | B^+(0) = x\} = \frac{\Phi\left(\frac{y+x}{\sqrt{t}}\right) - \Phi\left(\frac{y-x}{\sqrt{t}}\right)}{\Phi\left(\frac{x}{\sqrt{t}}\right) - \Phi\left(\frac{-x}{\sqrt{t}}\right)}$$

Brownian Motion with drift (8.34) p.419

$X(t) = \mu t + \sigma B(t), \quad \text{for} \quad t \geq 0$

Absorption probabilities (8.40) p.420

$$u(x) = \mathbb{P}\left(X(T) = b | X(0) = x\right) = \frac{\exp(-2\mu x/\sigma^2) - \exp(-2\mu a/\sigma^2)}{\exp(-2\mu b/\sigma^2) - \exp(-2\mu a/\sigma^2)}.$$

Mean time to absorption (8.42) p.421

$v(x) = \mathbb{E}\left(T | X(0) = x\right) = 1/\mu\left(u(x)(b-a) - (x-a)\right).$

Geometric Brownian motion (8.49) p.424

$Z(t) = e^{X(t)} = ze^{\mu t + \sigma B(t)} = ze^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma B(t)}$

Mean of geometric Brownian motion (8.50) p.425

$E\left[Z(t) | Z(0) = z\right] = ze^{\mu t} e^{\frac{1}{2}\sigma^2 t} = ze^{(\mu + \frac{1}{2}\sigma^2)t} = ze^{\alpha t}$

Absorption probabilities (8.52) p.426

$$T = T_{A,b} = \min\left\{t \geq 0 : Z(t) \in \{Z(0)A, Z(0)B\}\right\}, \quad \Pr\left\{Z(T) = Z(0)B\right\} = \frac{1 - A^{1-2\alpha/\sigma^2}}{B^{1-2\alpha/\sigma^2} - A^{1-2\alpha/\sigma^2}}$$

Some typos

Problem 8.4.4 $\frac{1}{2}$ should be $\frac{t}{2}$.