Brownian motion Section 8.1.2 P.392 Solution of diffusion equation

\[ \frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2} \]

With \( M_t(\theta) = E(\exp(-\theta X(t))) \) provided such \( \theta \) exists to make the expectation exist. Analytically the expectation is the Laplace transform of the density, and thus the transform of the second derivative (for fixed \( t \)) is \( \int_{-\infty}^{\infty} \exp(-\theta y) \frac{\partial^2 p}{\partial y^2} dy = \theta^2 \int_{-\infty}^{\infty} \exp(-\theta y)p(y,t|x)dy = \theta^2 M_t(\theta) \) (by partial integration). Thus \( \frac{\partial M}{\partial t} = \frac{1}{2} \theta^2 M_t(\theta) \Rightarrow M_t(\theta) = \exp\left(\frac{\theta^2}{2}\right) \), thus \( X(t) \sim N(0,t) \).

Invariance Principle Section 8.1.3 P. 396

Gaussian Process Section 8.1.4 P.398

Maximum in finite interval - equivalently - time to first reach a level Sections 8.2.1-8.2.2 P.406

Zeros of Brownian Motion Section 8.2.3 P.408
Some important definitions and results

Transition density continuous time Markov process (8.1)   p.322
Pr \{X(t) \in dy | X(0) = x\} = p(y,t|x)

Diffusion equation (8.3) with solution (8.4)   p.392
\[ \frac{\partial p}{\partial t} = \frac{1}{2} \sigma^2 \frac{\partial^2 p}{\partial y^2} \quad p(y,t|x) = \frac{1}{\sqrt{2\pi t\sigma}} e^{-\frac{(y-x)^2}{2\sigma^2 t}} \]

Definition of Brownian motion   p.394
\{B(t); t \geq 0\} with independent increments
\[ B(t_n) - B(t_{n-1}) \sim N\left(0, \sigma^2(t_n-t_{n-1})\right), \quad X(0) = 0 \]

Covariance function   p.396
\[ \text{Cov}(B(s), B(t)) = \sigma^2 \min\{s, t\} \]

Invariance principle (central limit theorem)   p.396
\[ B_n(t) = \frac{S_{\left\lfloor nt \right\rfloor}}{\sqrt{n}}, \quad B_n(t) = \frac{\sqrt{\left\lfloor nt \right\rfloor}}{\sqrt{k}} \frac{S_k}{\sqrt{k}} \quad \text{for } n \to \infty \]

Definition of multivariate Gaussian distribution   p.398
\[ \mathbf{X} = (X_1, \ldots, X_n) \in N(\mu, \Gamma) \text{ if } \sum_{j=1}^n \alpha_j X_j \in N(\alpha \mu, \alpha \Gamma) \quad \forall \alpha \in \mathbb{R}^n, \quad f(x) = (2\pi)^{-\frac{n}{2}} \det(\Gamma)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu)'\Gamma^{-1}(x-\mu)} \]

Reflection around \( \tau \) with \( B(\tau) = x \)   p.406
\[ B^*(u) = \begin{cases} B(u) & \text{for } u \leq \tau \\ x - [B(u) - x] & \text{for } u > \tau \end{cases} \]

Maximum \( M(t) \) of process up to time \( t \) (8.19) and (8.20)   p.407
\[ M(t) = \max_{0 \leq u \leq t} B(u), \quad \Pr[M(t) > x] = 2[1 - \Phi(x/\sqrt{t})] \]

Distribution of first hitting time \( \tau_x \) at \( x \) (8.21), (8.22), and (8.23)   p.407
\[ \tau_x = \min\{u \geq 0 : B(u) = x\}, \quad \Pr[\tau_x \leq ] = \frac{2}{\pi} \int_x^\infty e^{-\frac{z^2}{2}} \text{d}z = \sqrt{\frac{2}{\pi t}} \int_x^\infty e^{-\frac{z^2}{2t}} \text{d}z \]

Probability that Brownian motion reaches 0 between \( t \) and \( t + s \) (8.25)   p.407
\[ \Pr[\exists u \in (t, t+s) : B(u) = 0] = \frac{2}{\pi} \arctan\left(\frac{s}{t}\right) = \frac{2}{\pi} \arccos\left(\sqrt{\frac{t}{t+s}}\right) \]