Week 10

02407 Stochastic Processes
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BFN/bfn

Brownian motion Section 8.1.2 P.392

The litterature on Brownian motion is huge demonstrating its importance. The process can be understood in various ways. Like the Poisson Process, Brownian motion has independent increments (P.394) and they are both special cases of Levy processes, the most general stochastic process with independent and identically distributed increments. The diffusion equation (8.3), first derived by Einstein using physical arguments, can also be motivated through a symmetric simple random walk by a proper rescaling of time and space. I have not covered Section 2.5 on martingales but you have been exposed to a few exercises introducing the concept. Martingales constitute a broad class of stochastic processes similar to Markov processes. Many Markov processes are or can easily be written expressed through martingales leading to easy and elegant derivations of absorption probabilities and distributions for first passage times. Stochastic integration is defined in terms of (semi)martingales as we will touch very briefly in Section 8.5.4. The textbook introduces Brownian motion as a continuous time Markov process with a transition kernel that solves the diffusion equation (8.3)

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}.$$

We can solve the equation using a transform approach with $M_t(\theta) = \mathbb{E}\left(\exp\left(-i\theta X(t)\right)\right)$ being the characteristic function. Analytically the expectation is the Fourier transform of the density, and thus the transform of the second derivative (for fixed t) is $\int_{-\infty}^{\infty} \exp\left(-i\theta y\right)\frac{\partial^2 p}{\partial y^2}\mathrm{d}y = -\theta^2 \int_{-\infty}^{\infty} \exp\left(-i\theta y\right)p(y,t|x)\mathrm{d}y = -\theta^2 M_t(\theta)$ (by partial integration and p vanishing at infinity). So $\frac{\partial M}{\partial t} = -\frac{1}{2}\theta^2 M_t(\theta) \Rightarrow M_t(\theta) = \exp\left(-\theta^2 t/2\right)$, thus $X(t) \sim \mathrm{N}(0,t)$.

Invariance Principle Section 8.1.3 P. 396

For a random walk $S_n = \sum_{i=1}^{n} Z_i$, Z_i iid, define $B_n(t) = \frac{S_{[nt]}}{\sqrt{n}}$, where $[nt]$ is the largest integer not exceeding nt. This process is piecewise deterministic and asymptotically $B_n(t)$ is normal due to the central limit theorem with independent increments. As a consequence, Brownian motion is a good approximation to random walks for large n. The principle can be applied both ways in theoretical derivations as well as for examples. The test statistic in the Kolmogorov Smirnov goodness of fit test is based on the invariance principle (Donsker's theorem)and the Brownian bridge in Section 8.3.3.

Gaussian Process Section 8.1.4 P.398

A Gaussain process has finite dimensional distributions that are multivariate Gaussian. They appear in Brownian motion and its variants and are used extensively in Bayesian statistics and machine learning as a second order method, i.e. fitting the mean field and covariance structure. Gaussian processes can be used to express uncertainty in e.g. Krieging. The wikipedia page gives a fine brief introduction with several references.

Maximum in finite interval - equivalently - time to first reach a level Sections 8.2.1-8.2.2 P.406

The reflection principle - Section 8.2.1 - along with the strong Markov property is invoked to derive the distribution of the maximum. The maximum of a Brownian motion is distributionally equivalent to the first passage time, a connection similar to the one used to define the Erlang distribution as the distribution of a first passage time in the Poisson process. These two sections are a continuation of our previous studies of absorption and first passage time problems.

Zeros of Brownian Motion Section 8.2.3 P.408

Must read and nice to read

Section 8.1 and Section 8.2 are tightly written. Only the example *Cable Strength Under Equal Load Sharing* starting Page 399, can be read cursorily.

Some important definitions and results

Transition density continuous time Markov process (8.1) p.322

$\Pr\{X(t) \in \mathrm{d}y | X(0) = x\} = p(y,t|x)\mathrm{d}y$

Diffusion equation (8.3) with solution (8.4) p.392

$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2} \quad p(y,t|x) = \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{(y-x)^2}{2\sigma^2 t}}$

Definition of Brownian motion p.394

$\{B(t); t \geq 0\}$ with independent increments $B(t_n) - B(t_{n-1}) \sim \mathrm{N}\left(0, \sigma^2 (t_n - t_{n-1})\right), \quad X(0) = 0$

Covariance function p.396

$\mathrm{Cov}(B(s), B(t) = \sigma^2 \min\{s,t\}$

Invariance principle (cental limit theorem) p.396

$B_n(t) = \frac{S_{[nt]}}{\sqrt{n}}, \quad B_n(t) = \frac{\sqrt{[nt]}}{\sqrt{n}} \frac{S_k}{\sqrt{k}}, \frac{k}{n} \leq t < \frac{k}{n} + \frac{1}{n} \to B(t)$ for $n \to \infty$

Definition of multivariate Gaussian distribution p.398

$\boldsymbol{X} = (X_1, \dots, X_n) \in \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Gamma})$ if $\sum_{j=1}^{n} \alpha_j X_j \in \mathrm{N}(\boldsymbol{\alpha\mu}, \boldsymbol{\alpha\Gamma\alpha}') \; \forall \boldsymbol{\alpha} \in \mathbb{R}^n, \quad f(\boldsymbol{x}) = (2\pi)^{-\frac{n}{2}} \mathrm{Det}(\boldsymbol{\Gamma})^{-\frac{1}{2}}) e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Gamma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$

Reflection around τ with $B(\tau) = x$ p.406

$B^*(u) = \begin{cases} B(u) & \text{for } u \leq \tau \\ x - [B(u) - x] & \end{cases}$

Maximum $M(t)$ of process up to time t (8.19) and (8.20) p.407

$M(t) = \max_{0 \leq u \leq t} B(u), \quad \Pr\{M(t) > x\} = 2[1 - \Phi(x/\sqrt{t})]$

Distribution of first hitting time τ_x at x (8.21), (8.22), and (8.23) p.407

$\tau_x = \min\{u \geq 0 : B(u) = x\}, \quad \Pr\{\tau_x \leq\} = 2[1 - \Phi(x/\sqrt{t})] = \sqrt{\frac{2}{\pi t}} \int_x^\infty e^{-\frac{z^2}{2t}} \mathrm{d}z = \sqrt{\frac{2}{\pi t}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{-\frac{z^2}{2}} \mathrm{d}z$

Probability that Brownian motion reaches 0 between t and $t+s$ (8.25) p.407

$\Pr\{\exists u \in (t, t+s[: B(u) = 0\} = \frac{2}{\pi} \arctan\left(\frac{s}{t}\right) = \frac{2}{\pi} \arccos\left(\sqrt{\frac{t}{t+s}}\right)$

Some typos

Figure 8.1 (Page 395), Figure 8.4 (Page 406), Figure 8.5 (Page 413): The curves are not true functions, i.e. for some values of x we have multiple values.