Let $\{N_t\}_{t\geq 0}$ denote an renewal process with interarrival times T_i, $i \in \mathbb{N}$, i.i.d. according to the Laplace transform $\mathcal{L}(s)$ as given in the exercise.

a) We can derive the non-central moments of the interarrival times directly from the Laplace transform. Specifically,

$$\mathbb{E}[T_i^n] = (-1)^n L^{(n)}(s)|_0 .$$

Hence

$$\mathbb{E}[T_i] = -L'(0) = 1,$$
$$\mathbb{E}[T_i^2] = L''(0) = 5/3,$$
$$\mathbb{V}[T_i] = \mathbb{E}[T_i^2] - \mathbb{E}[T_i]^2 = 5/3 - 1^2 = 2/3.$$

b) We could find the exact solution to this problem, but it would require us to find the convolution $\sum_{i=1}^{30} T_i$. Instead, we will apply the limit result for $N(t)$ and give an approximate result. In sec. 7.4.3, we are given the asymptotic result that

$$N(t) \sim \mathcal{N}(t/\mu, t\sigma^2/\mu^3).$$

Thus, for $t = 24$, $N(24) \sim \mathcal{N}(24, 16)$.

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By standardization, $[N(24)-24]/\sqrt{16}$ follows a standard normal distribution.

Hence,

$$
\begin{aligned}
\mathbb{P}(N(24) \geq 30) &= 1 - \mathbb{P}(N(24) < 30) \\
&= 1 - \mathbb{P}(N(24)^* < (30-24)/4) \\
&= 1 - \mathbb{P}(N(24)^* < 3/2) \\
&= 1 - \Phi(3/2),
\end{aligned}
$$

where * denotes a standardized variable and Φ is the CDF of the standard normal distribution. Since we are approximating, it might be appropriate to invoke a continuity correction as in Wolff (1989), sec. 2.20. In that case, we would get:

$$
\begin{aligned}
\mathbb{P}(N(24) \geq 30) &= 1 - \mathbb{P}(N(24)^* < (30 - 1/2 - 24)/4) \\
&= 1 - \Phi(11/8).
\end{aligned}
$$

c) We shall find $\mathbb{E}[N_t] = M_t$ through the Laplace transform of M_t. From the note on MGFs and Laplace transforms, we get that $\mathcal{L}_{M_t}(s) = \int_0^\infty e^{-st} M_t \, dt$ (recall that M_t is a function!). can be given as

$$\mathcal{L}_{M_t}(s) = \mathcal{L}_{T_i}(s) / [s(1 - \mathcal{L}_{T_i}(s))].$$

By rather tedious and trivial calculations, we get that

$$\mathcal{L}_{M_t}(s) = \frac{1}{s^2} - \frac{1}{6s} + \frac{1}{6}\frac{(s+7/6)}{(s+7/6)^2+23/36} - \frac{1}{6}\frac{5/6}{(s+7/6)^2+23/36}.$$

We then get M_t as the inverse Laplace transform of $\mathcal{L}_{M_t}(s)$:

$$M_t = \mathcal{L}^{-1}(\mathcal{L}_{M_t}(s))(t)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{6s} + \frac{1}{6}\cdot\frac{(s+7/6)}{(s+7/6)^2+23/36} - \frac{1}{6}\cdot\frac{5/6}{(s+7/6)^2+23/36}\right)(t).$$

The inverse Laplace is linear and we can transform each term seperately.

$$M_t = t - \frac{1}{6} + \frac{1}{6}e^{-7/6\cdot t}\cos\left(\sqrt{23/36}\,t\right) - \frac{1}{6}\cdot\frac{5}{6}\cdot\sqrt{36/23}\,e^{-7/6\cdot t}\sin\left(\sqrt{23/36}\,t\right)$$

for $t \geq 0$. We can simplify this sligthly as

$$M_t = t - \frac{1}{6} + \frac{1}{6}e^{-7/6\cdot t}\left(\cos\left(\sqrt{23/36}\,t\right) - 5/\sqrt{23}\sin\left(\sqrt{23/36}\,t\right)\right).$$

d) We shall now find the distribution of the residual lifetime as defined on p. 349 in KP. Thus, let γ_t denote the residual lifetime. From sec. 7.4.4 in KP, we know the limiting distribution:

$$\lim_{t\to\infty} \mathbb{P}(\gamma_t \leq x) = \mu^{-1}\int_0^x 1 - F(y)\,dy, \qquad (7.21)$$

where F is the CDF of T_i.

Consequently, asymptotically we have

$$F_{\gamma_t}(x) = H(x) \approx \mu^{-1}\int_0^x 1 - F(y)dy.$$

Taking the derivative wrt. x on both sides yields

$$f_{\gamma_t}(x) = H'(x) \approx \mu^{-1}(1 - F(x)).$$

Recall that we can obtain the CDF from the Laplace transform via

$$F(x) = \mathcal{L}^{-1}(\mathcal{L}(s)/s)(x).$$

You can now use partial fractions decomposition to get an expression, which is easy to invert. I get

$$\mathcal{L}(s)/s = \frac{1}{s} - \frac{2}{3}\cdot\frac{1}{(s+1)} - \frac{1}{3}\cdot\frac{(s+2)}{(s+1)^2+1}$$

$$= \frac{1}{s} - \frac{2}{3}\cdot\frac{1}{(s+1)} - \frac{1}{3}\left(\frac{1}{(s+1)^2+1} + \frac{(s+1)}{(s+1)^2+1}\right).$$

Hence,

$$F(x) = 1 - \frac{2}{3}e^{-t} - \frac{1}{3}\left(e^{-t}\sin(t) + e^{-t}\cos(t)\right)$$

$$= 1 - \frac{1}{3}e^{-t}(2 + \sin(t) + \cos(t)).$$

In conclusion,

$$f_{\gamma_t}(x) = \mu^{-1}(1 - F(x)) = \frac{1}{3}e^{-t}(2 + \sin(t) + \cos(t)).$$

e) In the last problem (d.), we found that the remaining lifetime had the distribution function (CDF) $H(x)$.

Note also that γ_t (the residual lifetime) is the first lifetime after we lost the information on the time in service of the component in service.

Since the first lifetime has distribution function:

$$H(x) = \mu^{-1} \int_0^x 1 - F(y)\, dy,$$

the renewal process has reached stationarity, cf. sec. 7.5.2. Under stationarity, we can apply eq. (7.26)

$$M_D(t) = \mathbb{E}[N_t] = t/\mu = t.$$