Exercise 14

Q1 We consider the life time of an electronic piece of equipment. It can be in four different states:

3: OK
2: Uncertain
1: Critical
0: Defect

Given the information in the problem, the equipment will change condition according to the generator

$$Q = \begin{array}{c} 3 \\ 2 \\ 1 \\ 0 \end{array} \begin{bmatrix} -1/100 & 95/100 \cdot 1/100 & 5/100 \cdot 1/100 & 0 \\ 8/10 \cdot 1/10 & -1/10 & 2/10 \cdot 1/10 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We let $\{X_t\}_{t \geq 0}$ be a CTMC governed by Q, i.e. X_t describes the condition of the equipment at time $t \geq 0$. Now, define T as the life time of the equipment. Formally,

$$T = \inf\{t \geq 0 : X_t = 0\}.$$

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We choose to model T with a PH distribution, namely $T \sim PH(\alpha, S)$, where $\alpha = (1, 0, 0)$ (it always starts in OK condition) and subgenerator

$$S = \begin{pmatrix} -1/100 & 95/100^2 & 5/100^2 \\ 8/100 & -1/10 & 2/100 \\ 0 & 0 & -1 \end{pmatrix}.$$

Q2 We know from the exit rate vector

$$s = -S\,1_{3\times 1} = \begin{array}{c} 3 \\ 2 \\ 1 \end{array}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

that the process can only be absorbed from state 1. Consequently, the process will always visit state 1 before being absorbed. Moreover, we know that the process always will be absorbed after visiting state 1. Therefore, we (the process) always visit state 1 exactly once. In conclusion, the time spent in state 1, say T_1, is exponentially distributed with rate 1. Thus,

$$\mathbb{P}(T_1 > 2) = e^{-2}.$$

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Q3 The new replacement policy makes $\{X_t\}$ a PH-renewal process, where the interarrival times are distributed identically to T from Q1. Assuming the process has reached stationarity, we can apply Th. 6 from the reading note:

$$\pi = \alpha(-S)^{-1} / \alpha(-S)^{-1} 1_{3\times1} = \left({}^{5000}/_{5487}, {}^{475}/_{5487}, {}^{12}/_{5487} \right).$$

Hence $\pi_{OK} = \pi_3 = {}^{5000}/_{5487}$.

Q4 We are given that $\pi(S + s\alpha) = 0_{1\times3}$. As S is a subgenerator, we know that it is non-singular and therefore invertible.

Thus, we can multiply both sides of $\pi(S+s\alpha) = 0_{1\times3}$ by $S^{-1}1_{3\times1}$. Hence

$$\pi(S+s\alpha)S^{-1}1_{3\times1} = \pi 1_{3\times1} + \pi s \alpha S^{-1} 1_{3\times1} = 0.$$

Note that $\pi 1_{3\times1} = 1$ as the elements of π has to sum to one. Furthermore, from Cor. 1 in the Reading Note, we know that $\mu = \alpha(-S)^{-1}1_{3\times1}$. So the LHS of the above simplifies to

$$1 - \pi s \mu = 0 \implies (\pi s)^{-1} = \mu.$$

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Q5 From Q3, we know π, and in Q4, we just derived that $\mu = (\pi S)^{-1}$. So,

$$\mu = (\pi S)^{-1} = \left(\left(5000/5487,\ 475/5487,\ 12/5487 \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)^{-1}$$

$$= \left(12/5487 \right)^{-1} = 5487/12.$$