Exercise 1

Before formulating specific models, we need to establish some assumptions:

- Independence between the passing couples
- Independence between males and females in each and all couples.
- Identical behaviour for all couples, males and females.

Q1 In each couple, we assume that the male has probability p of buying an ice cream when passing the parlour.

Let T_M denote the number of couples who need to pass before a male want to buy an ice cream. Then $T_M \sim DPH(\alpha_M, S_M)$, where $\alpha_M = \{1\}$ and $S_M = \{1-p\}$, i.e. $T_M \sim Geo(p)$.

Q2 In a similar manner, we can define T_F and model $T_F \sim DPH(\alpha_F, S_F)$, where $\alpha_F = \{1\}$, $S_F = \{1-q\}$ such that $S_F \sim Geo(q)$.

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We then define T_c as the number of couples that need to pass before a couple buys at least one ice cream.

We realize that $T_c = \min(T_M, T_F)$ and apply Th. 1.2.67 from Bladt and Nielsen (2017), which we shall abbreviate BN2017.

Hence,

$T_c \sim DPH(\alpha_M \otimes \alpha_F, S_M \otimes S_F)$,

where $\otimes$ denotes the Kronecker product. Thus, $\alpha_M \otimes \alpha_F = 1 \cdot 1 = 1$ and, $S_M \otimes S_F = (1-p)(1-q)$, so $\alpha_c = \{1\}$ and $S_c = \{(1-p)(1-q)\}$, which means T_c is also geometrically distributed as $T_c \sim Geo(p+q-pq)$.

Q3 We know from Q2 that each couple has probability $p+q-pq$ of buying at least one ice cream, and since we assumed independence, the number of couples who need to pass the parlour before two couples have bought at least one ice cream, say T_{2c}, must follow an $NB(2, p+q-pq)$ distribution, i.e. negative binomial.

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In terms of DPH distributions, T_{2c} can be considered as a sum of two independent random variables with the same distribution as T_c. You can think of it as first waiting for the first couple to buy at least one ice cream and afterwards waiting for an additional couple to buy at least one ice cream. Thus, we can apply Th. 1.2.65 from BN2017. This yields that $T_{2c} \sim DPH(\alpha_{2c}, S_{2c})$, where

$$\alpha_{2c} = (\alpha_c \;\; 0) = (1, 0),$$

$$S_{2c} = \begin{pmatrix} S_c & s_c \\ 0 & S_c \end{pmatrix} = \begin{pmatrix} (1-p)(1-q) & 1-(1-p)(1-q) \\ 0 & (1-p)(1-q) \end{pmatrix}.$$

Here s_c (not S_c) is given by $s_c = 1 - S_c$, cf. p. 29 in BN2017.

Q4 Recall the definitions of T_M and T_F. The number of couples that need to pass before an ice cream has been bought by a person of each sex must be $\max(T_M, T_F)$. Either the male buys first ($T_M < T_F$) and when the female buys, they have both bought, or the female buys first ($T_F < T_M$) and when the male buys, they have both bought.

Exercise 1

Finally, it may happen that $T_M = T_F$, i.e. the first male buyer is in a relationship with the first female buyer. However, in all cases, we seek $\max(T_M, T_F)$.

Applying again Th. 1.2.67, we get that

$T_{MF} = \max(T_M, T_F) \sim DPH(\alpha_{MF}, S_{MF})$, where

$\alpha_{MF} = (\alpha_M \otimes \alpha_F, 0, 0) = (1, 0, 0)$, and

$$S_{MF} = \begin{pmatrix} S_M \otimes S_F & S_M \otimes s_F & s_M \otimes S_F \\ 0 & S_M & 0 \\ 0 & 0 & S_F \end{pmatrix}$$

$$= \begin{array}{c} MF \\ M \\ F \end{array} \begin{pmatrix} (1-p)(1-q) & (1-p)q & p(1-q) \\ 0 & (1-p) & 0 \\ 0 & 0 & (1-q) \end{pmatrix}.$$

You can think of the states is S_{MF} as:

MF: Both have not bought an ice cream.

M: At least one female has bought an ice cream, but no male has bought one

F: At least one male has bought an ice cream, but no female has bought one.