# Problem 6.5.2

We consider a birth and death process governed by the generator

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -5 & 1 & 0 & 0 & 0 \\ 0 & 3 & -5 & 2 & 0 & 0 \\ 0 & 0 & 2 & -5 & 3 & 0 \\ 0 & 0 & 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) You may use the formula on the top of p. 318, but you need to argue why this is allowed considering the different setups in problem and in the text, specifically we have a case with a finite state-space and a case with an infinite state-space.

I prefer to circumvent that and simply apply a "First step analysis". We let $u_i$ denote the probability of being absorbed in state zero given initiation in state $i$.

We can then formulate a linear system whose solution yields the desired probability:

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$u_1 = 4/5 + 1/5\, u_2,$

$u_2 = 3/5\, u_1 + 2/5\, u_3,$

$u_3 = 2/5\, u_2 + 3/5\, u_4,$

$u_4 = 1/5\, u_3,$

Note that the transition probabilities arise from the embedded Markov chain, i.e. $p_i = \lambda_i/(\lambda_i + \mu_i)$ and $q_i = 1 - p_i$.

The system yields the solution:

$(u_1, u_2, u_3, u_4) = (15/16, 11/16, 5/16, 1/16)$, so $u_2 = 11/16$.

b) We could simply use eq. (6.45) to solve the problem, but since the topic of the week is phase-type distributions, we shall use such a distribution. In order to do so, we need to merge the absorping states and modify the generator accordingly.

$$Q' = \begin{array}{c} 0+5 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & -5 & 1 & 0 & 0 \\ 0 & 3 & -5 & 2 & 0 \\ 0 & 0 & 2 & -5 & 3 \\ 4 & 0 & 0 & 1 & -5 \end{bmatrix}$$

We can now define a random variable $T$ as the time until absorption given the process is initiated in state 2. For the sake of convention, we flip the generator such that the absorping

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state is in the bottom, i.e.

$$Q' = \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \\ 0+5 \end{array} \begin{bmatrix} -5 & 1 & 0 & 0 & 4 \\ 3 & -5 & 2 & 0 & 0 \\ 0 & 2 & -5 & 3 & 0 \\ 0 & 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $T \sim PH(\alpha, S)$, where $\alpha = (0,0,1,0)$ and $S$ consists of rows and columns 1 to 4 of $Q'$. Therefore, we find that

$$\mathbb{E}[T] = \alpha(-S)^{-1} \mathbb{1}_{4\times1}, \qquad \mathbb{1}_{4\times1} = (1,1,1,1)^T.$$

Hence $\mathbb{E}[T] = 2/3$.