Exercise 13

We know that the lifetime of the electronic component, say T, has a phase-type distribution, specifically $T \sim PH(\alpha, S)$, where

$$\alpha^* = p(0) = (3/4, 1/4, 0), \quad \alpha = (3/4, 1/4)$$

$$S = \begin{pmatrix} -1 & 9/10 \\ 12/55 & -12/55 \end{pmatrix}.$$

Q1 First we need to find the expectation and the variance of T. To find the expected value, we apply Th. 3.1.16 in BN2017, i.e.

$$\mathbb{E}[T] = \alpha(-S)^{-1} 1_{2\times1} = (3/4, 1/4)\begin{pmatrix} 10 & \frac{165}{4} \\ 10 & \frac{275}{6} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2515}{48}.$$

The second moment is found using Cor. 3.1.18 in BN2017

$$\mathbb{E}[T^2] = 2!\alpha(-S)^{-2} 1_{2\times1} = 829325/144.$$

Hence

$$\mathbb{V}[T] = \mathbb{E}[T^2] - \mathbb{E}[T]^2 = 6943975/2304.$$

Exercise 13

Q2 You are allowed to use the formulas in BN2017 and let Maple or similar software calculate the expression. If you insist on doing by hand, you can use the Laplace transform to your advantage. Applying Th. 3.1.19 in BN2017 yields

$$\mathcal{L}(s) = \alpha(sI - S)^{-1}(-S\mathbf{1}_{2\times1}) = \frac{5}{104}\left(\frac{6/5}{s+6/5}\right) + \frac{99}{104}\left(\frac{1/55}{s+1/55}\right)$$

$$= \frac{6}{104}(s+6/5)^{-1} + \frac{9}{520}(s+1/55)^{-1}.$$

The probability density function can now be recovered from the Laplace transform through the inverse transform

$$f_T(t) = \mathcal{L}^{-1}\left\{\frac{6}{104}(s+6/5)^{-1} + \frac{9}{520}(s+1/55)^{-1}\right\}(t)$$

$$= \frac{6}{104}\mathcal{L}^{-1}\{(s+6/5)^{-1}\}(t) + \frac{9}{520}\mathcal{L}^{-1}\{(s+1/55)^{-1}\}(t)$$

$$= \frac{3}{52}e^{-\frac{6}{5}t} + \frac{9}{520}e^{-\frac{1}{55}t}, \quad t > 0.$$

Exercise 13

Q3 We now consider a PH renewal process, which is an arrival process with independent PH-distributed interarrival times.

Assuming the process has reached stationarity, we can find the stationary distribution using Th. 6 in the reading note. Hence, for $U = (-S)^{-1}$,

$$\pi = \alpha U / \alpha U \mathbb{1}_{2\times 1}$$

$$= \begin{pmatrix} 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 10 & \frac{165}{4} \\ 10 & \frac{275}{6} \end{pmatrix} \Big/ \begin{pmatrix} 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 10 & \frac{165}{4} \\ 10 & \frac{275}{6} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 96/503 & 407/503 \end{pmatrix}.$$

Q4 The distribution of the remaining lifetime, assuming the renewal process has reached stationarity, is given by a $PH(\pi, S)$-distribution. We can calculate the moments like we did in Q1: Let T_r denote the remaining lifetime, such that $T_r \sim PH(\pi, S)$. Then

$$\mathbb{E}[T_r] = \cancel{694}\ 165865/3018$$

$$\mathbb{E}[T_r^2] = 54734825/9054$$

$$\mathbb{V}[T_r] = \mathbb{E}[T_r^2] - \mathbb{E}[T_r]^2 = 27552035725/9108324.$$