Exercise 6.6.2

Let $\{Z_t\}_{t\geq 0}$ be a CTMC given by $Z_t = X_t^1 + X_t^2$. Here $\{X_t^1\}_{t\geq 0}$ and $\{X_t^2\}_{t\geq 0}$ are independent CTMCs with the common generator:

$$A = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

From the example on p. 329-331, we know the transition probability matrix $P^*(t)$ for $\{X_t^1\}$ and $\{X_t^2\}$.

$$P^*(t) = \begin{pmatrix} \mu/(\lambda+\mu) & \lambda/(\lambda+\mu) \\ \mu/(\lambda+\mu) & \lambda/(\lambda+\mu) \end{pmatrix} + \begin{pmatrix} \lambda/(\lambda+\mu) & -\lambda/(\lambda+\mu) \\ -\mu/(\lambda+\mu) & \mu/(\lambda+\mu) \end{pmatrix} e^{-(\lambda+\mu)t}$$

To construct $P(t)$, we note that

$\{Z_t = 2 \mid Z_0 = 0\} = \{X_t^1 = 1, X_t^2 = 1 \mid X_0^1 = 0, X_0^2 = 0\}$,

$\{Z_t = 1 \mid Z_0 = 0\} = \{X_t^1 = 1, X_t^2 = 0 \mid X_0^1 = 0, X_0^2 = 0\}$
$\cup \{X_t^1 = 0, X_t^2 = 1 \mid X_0^1 = 0, X_0^2 = 0\}$, and

$\{Z_t = 0 \mid Z_0 = 0\} = \{X_t^1 = 0, X_t^2 = 0 \mid X_0^1 = 0, X_0^2 = 0\}$.

Therefore,

$$\begin{aligned} P_{02}(t) &= \mathbb{P}(Z_t = 2 \mid Z_0 = 0) \\ &= \mathbb{P}(X_t^1 = 1, X_t^2 = 1 \mid X_0^1 = 0, X_0^2 = 0) \\ \text{(indep.)} &= \mathbb{P}(X_t^1 = 1 \mid X_0^1 = 0)\,\mathbb{P}(X_t^2 = 1 \mid X_0^2 = 0) \\ &= P_{01}^*(t) P_{01}^*(t) = P_{01}^*(t)^2. \end{aligned}$$

Exercise 6.6.2

Similarly,

$$\begin{aligned}
P_{01}(t) &= \mathbb{P}(Z_t = 1 \mid Z_0 = 0) \\
&= \mathbb{P}(X_t^1 = 1, X_t^2 = 0 \mid X_0^1 = 0, X_0^2 = 0) \\
&\quad + \mathbb{P}(X_t^1 = 0, X_t^2 = 1 \mid X_0^1 = 0, X_0^2 = 0) \\
\text{(indep.)} \quad &= \mathbb{P}(X_t^1 = 1 \mid X_0^1 = 0)\,\mathbb{P}(X_t^2 = 0 \mid X_0^2 = 0) \\
&\quad + \mathbb{P}(X_t^1 = 0 \mid X_0^1 = 0)\,\mathbb{P}(X_t^2 = 1 \mid X_0^2 = 0) \\
&= P_{01}^*(t) P_{00}^*(t) + P_{00}^*(t) P_{01}^*(t) \\
&= 2 P_{00}^*(t) P_{01}^*(t).
\end{aligned}$$

Finally,

$$P_{00}(t) = P_{00}^*(t)^2.$$

Collecting all the terms and putting them into a matrix:

$$P(t) = \begin{pmatrix} P_{00}^*(t)^2 & 2 P_{00}^*(t) P_{01}^*(t) & P_{01}^*(t)^2 \\ 2 P_{10}^*(t) P_{00}^*(t) & 2\left(P_{00}^*(t) P_{11}^*(t) + P_{10}^*(t) P_{01}^*(t)\right) & 2 P_{01}^*(t) P_{11}^*(t) \\ P_{10}^*(t)^2 & 2 P_{10}^*(t) P_{11}^*(t) & P_{11}^*(t)^2 \end{pmatrix}$$

We shall not write out all elements, but we shall give an example.

$$P_{00}(t) = P_{00}^*(t)^2 = \left(\mu/(\lambda+\mu) + \lambda/(\lambda+\mu)\, e^{-(\mu+\lambda)t} \right)^2.$$