

Exercise 27

1/4

We shall first develop a model.
We employ a CTMC $\{X_t\}_{t \geq 0}$ governed by the generator

$$Q = \begin{matrix} & \begin{matrix} I & J & JB \end{matrix} \\ \begin{matrix} I \\ J \\ JB \end{matrix} & \begin{pmatrix} -1 & 1 & 0 \\ 4 & -5 & 1 \\ 0 & 4 & -4 \end{pmatrix} \end{matrix}$$

This is a queueing system:
M/M/1/2,
 $\lambda = 1, \mu = 4$

Q1

In the long run, the fraction of customers who are lost is equal to the probability that an arriving customer is rejected. This probability must be the same as the probability that the process is in state JB. To see this, let R denote the event that an arriving customer is rejected. Then

$$\begin{aligned} P(R) &= P(R|X_t = I)P(X_t = I) + P(R|X_t = J)P(X_t = J) \\ &\quad + P(R|X_t = JB)P(X_t = JB) \\ &= P(X_t = JB) = \pi_{JB}. \end{aligned}$$

To find this, we apply eqs (6.36) + (6.37):

$$\theta_0 = 1, \theta_1 = \lambda/\mu = 1/4, \theta_2 = (\lambda/\mu)^2 = (1/4)^2 = 1/16$$

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This yields the stationary distribution:

$$(\pi_I, \pi_J, \pi_{JB}) = (16/21, 4/21, 1/21).$$

In conclusion, $\pi_{JB} = 1/21$.

Q2

The equipment is in use unless the repairman is idle. Thus, the probability of interest is

$$1 - (\pi_I) = 1 - 16/21 = 5/21.$$

Q3

We model the new scenario as follows:

$$Q = \begin{matrix} & \begin{matrix} I & J & R & JB & RB \end{matrix} \\ \begin{matrix} I \\ J \\ R \\ JB \\ RB \end{matrix} & \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 4 & -5 & 0 & 1 & 0 \\ 4 & 0 & -5 & 0 & 1 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & 4 & 0 & 0 & -4 \end{bmatrix} \end{matrix},$$

This allows us to calculate the transition probabilities as

$$P(t) = e^{Qt}. \quad (\text{See solution on the homepage for more info}).$$

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Q4

We find the stationary distribution by using eq. (6.68)

$$\underline{Q} = \Pi \underline{Q},$$

which yields

$$\begin{aligned} (\pi_I, \pi_J, \pi_R, \pi_{JB}, \pi_{RB}) &= (80/130, 24/130, 16/130, 6/130, 4/130) \\ &= (40/65, 12/65, 8/65, 3/65, 2/65). \end{aligned}$$

Q5

The equipment is utilized whenever the state is not "I" (Idle), i.e. the probability of interest is $1 - \pi_I = 5/13$.

Q6

I am not sure exactly what the question is asking. In all cases, we need the following components

STATE	RENTAL REQUESTS	WORK REQUESTS
I	REJECT: NO	REJECT: NO
J	REJECT: YES	REJECT: NO
R	REJECT: YES	REJECT: NO
JB	REJECT: YES	REJECT: YES
RB	REJECT: YES	REJECT: YES

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It is not clear to me exactly when customers are rejected because the equipment is not available, e.g. in state "RB".

We can however calculate the fraction of requests that are rejected. Since both types of requests arrive according to a Poisson arrival process with rate 1, every arrival has a 50/50 chance of being a work request or a rental request. Hence, the blocking probability is

$$\pi_{JB} + \pi_{RB} + \frac{1}{2}\pi_R + \frac{1}{2}\pi_J = \frac{6}{130} + \frac{4}{130} + \frac{1}{2} \left(\frac{16}{130} + \frac{24}{130} \right) \\ = \frac{30}{130} = \frac{3}{13}.$$