Exercise 6.6.2

Let $\{X_t^1\}_{t\geq 0}$ and $\{X_t^2\}_{t\geq 0}$ be independent CTMCs with the common generator A.

We then form $\{Z_t\}_{t\geq 0}$ as $Z_t = X_t^1 + X_t^2$. As $X_t^1, X_t^2 \in \{0,1\}$, the state space of $\{Z_t\}$ must be $S = \{0,1,2\}$. To obtain the generator, Q, of $\{Z_t\}$, we formally write:

$$q_{02} = \lim_{h\to 0} \mathbb{P}(Z_{t+h} = 2 \mid Z_t = 0)/h$$

$$= \lim_{h\to 0} \mathbb{P}(X_{t+h}^1 = 1,\ X_{t+h}^2 = 1 \mid X_t^1 = 0,\ X_t^2 = 0)/h.$$

Due to the independence of $\{X_t^1\}$ and $\{X_t^2\}$ we get:

$$q_{02} = \lim_{h\to 0} \mathbb{P}(X_{t+h}^1 = 1 \mid X_t^1 = 0)\mathbb{P}(X_{t+h}^2 = 1 \mid X_t^2 = 0)/h$$

$$= \lim_{h\to 0} (\lambda h + o(h))(\lambda h + o(h))/h$$

$$= \lim_{h\to 0} (\lambda^2 h^2 + 2h o(h) + o(h))/h$$

$$= \lim_{h\to 0} (\lambda^2 h + 2h\cdot o(h)/h + o(h)/h)$$

$$= 0.$$

Exercise 6.6.2

Similarly,

$$
\begin{aligned}
q_{01} &= \lim_{h\to 0} \mathbb{P}(Z_{t+h}=1 \mid Z_t=0)/h \\
&= \lim_{h\to 0} \mathbb{P}(X^1_{t+h}+X^2_{t+h}=1 \mid X^1_t+X^2_t=0)/h \\
&= \lim_{h\to 0} \mathbb{P}(X^1_{t+h}=1, X^2_{t+h}=0 \mid X^1_t=0, X^2_t=0)/h \\
&+ \lim_{h\to 0} \mathbb{P}(X^1_{t+h}=0, X^2_{t+h}=1 \mid X^1_t=0, X^2_t=0)/h \\
&= \lim_{h\to 0} \mathbb{P}(X^1_{t+h}=1 \mid X^1_t=0)\mathbb{P}(X^2_{t+h}=0 \mid X^2_t=0)/h \\
&+ \lim_{h\to 0} \mathbb{P}(X^1_{t+h}=0 \mid X^1_t=0)\mathbb{P}(X^2_{t+h}=1 \mid X^2_t=0)/h.
\end{aligned}
$$

Due to the symmetry of $\{X^1_t\}$ and $\{X^2_t\}$:

$$
\begin{aligned}
q_{01} &= 2\lim_{h\to 0} (\lambda h + o_1(h))(1-\lambda h + o_2(h))/h \\
&= 2\lim_{h\to 0} (\lambda h - (\lambda h)^2 + o(h))/h \\
&= 2\lambda.
\end{aligned}
$$

And thus, $q_{00} = -(q_{01}+q_{02}) = -2\lambda.$

Exercise 6.6.2

Continuing this way, we get

$$Q = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}.$$

We now find the transition probability matrix, $P(t)$, by using eq. (6.67). The formula is substantially easier to use, whenever Q is diagonalizable. Thus, we find the eigenvalues as the roots of the characteristic polynomial $p(x)$:

$$p(x) = \det(Q - xI)$$
$$= -x^3 - 3x^2(\lambda+\mu) - 2x(\lambda^2+\mu^2) - 4x\lambda\mu,$$

which yields the eigenvalues

$$x_1 = 0, \; x_2 = -(\lambda+\mu), \; x_3 = -2(\lambda+\mu)$$

All roots have algebraic multiplicity one, which implies that the associated eigenvectors have geometric multiplicity one. Consequently, the eigenvectors are linearly independent. By the diagonalization theorem we can conclude that Q is diagonalizable.

Exercise 6.6.2

Furthermore, the theorem states that

$$Q = S \Lambda S^{-1},$$

where S consists of the eigenvectors of Q, and $\Lambda = \text{diag}(x_0, x_1, x_2)$. Thus,

$$S = \begin{pmatrix} 1 & -\lambda/\mu & (\lambda/\mu)^2 \\ 1 & -(\mu+\lambda)/2\mu & -\lambda/\mu \\ 1 & 1 & 1 \end{pmatrix}. \quad \left(S_{22} = -(\lambda-\mu)/2\mu\right)$$

Hence,

$$P(t) = e^{Qt} = e^{S\Lambda t S^{-1}} = S e^{\Lambda t} S^{-1}.$$

Now, the matrix-exponential of a diagonal matrix is simply a diagonal matrix with the elements exponentiated, i.e.
$e^{\Lambda t} = \text{diag}(e^{x_0 t}, e^{x_1 t}, e^{x_2 t})$.

We shall not write out $P(t)$.