Problem 9.2.6

Let $\{X_t\}_{t\geq 0}$ be the number of customers in a M/M/1 queueing system with arrival rate λ and service rate μ.

We will consider the process after it has reached stationarity. Under this assumption, the probability of exceeding the capacity c is given by

$$\mathbb{P}(X_t > c) = 1 - \mathbb{P}(X_t \leq c) \qquad (c \in \mathbb{N})$$
$$= 1 - \sum_{i=0}^{c} \pi_i.$$

Using eq. (9.11) we rewrite this as

$$\mathbb{P}(X_t > c) = 1 - \sum_{i=0}^{c} (1-\rho)\rho^i,$$

where $\rho = \lambda/\mu$ is the traffic intensity. If we want this probability to be below $1/1000$, we need

$$\mathbb{P}(X_t > c) = 1 - \sum_{i=0}^{c} (1-\rho)\rho^i < 1/1000$$
$$\Leftrightarrow \sum_{i=0}^{c} (1-\rho)\rho^i > 1 - \frac{1}{1000}$$

Note that $(1-\rho)\rho^i = \rho^i - \rho^{i+1}$, which implies that $\sum_{i=0}^{c} (1-\rho)\rho^i = \sum_{i=0}^{c} (\rho^i - \rho^{i+1}) = 1 - \rho^{c+1}$.

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In conclusion, we need

$$1-\rho^{c+1} > 1 - 1/1000,$$

which leads to

$$\rho^{c+1} < \frac{1}{1000}.$$

Taking log on both sides and substracting one on both sides yields:

$$\log(\rho)(c+1) < \log(10^{-3})$$

$$\Leftrightarrow c+1 < \log(10^{-3})/\log(\rho)$$

$$\Leftrightarrow c < \log(10^{-3})/\log(\rho) - 1.$$