Problem 9.2.4

Let $\{X_t\}_{t\geq 0}$ be the number of customers in a M/M/1/N queueing system with arrival rate λ and service rate μ.

a) In this setting, $\{X_t\}$ is a birth and death process with the state space $E = \{0, \ldots, N\}$. Consequently, the parameters are given as

$$\lambda_0 = \lambda_1 = \ldots = \lambda_{N-1} = \lambda, \quad \mu_1 = \ldots = \mu_{N-1} = \mu_N = \mu,$$

and this completely determines the system.

b) To find the fraction of idle-time, we calculate the stationary distribution. We apply eq. (6.68):

$$\pi_0 \cdot \lambda = \pi_1 \cdot \mu,$$
$$\pi_1 \cdot \lambda = \pi_2 \cdot \mu,$$
$$\vdots$$
$$\pi_{N-1} \cdot \lambda = \pi_N \cdot \mu,$$
$$\pi_0 + \ldots + \pi_N = 1.$$

In general, we get $\pi_k = \pi_0 (\lambda/\mu)^k = \pi_0 \rho^k$.

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Thus,

$$\sum_{i=0}^{N} \pi_i = \sum_{i=0}^{N} \pi_0 \rho^i = \pi_0 \left(\sum_{i=0}^{N} \rho^i \right) = 1$$

leads to $\pi_0 = \left(\sum_{i=0}^{N} \rho^i \right)^{-1}$.

c) Due to the PASTA-theorem (Poisson Arrivals See Time Averages), the probability of an arriving customer being blocked (or lost) is simply the probability that $\{X_t\}$ is in state N, i.e. the blocking probability is π_N.