Exercise 6.4.4

Let $\{X_t\}_{t\geq 0}$ denote the number of operating machines at time $t \geq 0$. Then the process is of the birth and death type with the generator

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{bmatrix}.$$

a) To find π_0, we apply the formulas (6.35) to (6.37). We obtain the system.

$$\lambda \pi_0 = \mu \pi_1, \qquad (1)$$
$$(\mu+\lambda)\pi_1 = \lambda \pi_0 + 2\mu \pi_2,$$
$$2\mu \pi_2 = \lambda \pi_1, \qquad (2)$$
$$\pi_0 + \pi_1 + \pi_2 = 1. \qquad (3)$$

Eq. (1) yields $\pi_1 = (\lambda/\mu)\pi_0$, while eq. (2) gives $\pi_2 = (\lambda/2\mu)\pi_1 = (\lambda^2/2\mu^2)\pi_0$. Inserting in eq. (3) gives:

$$\pi_0 + \pi_1 + \pi_2 = \pi_0 + (\lambda/\mu)\pi_0 + (\lambda^2/2\mu^2)\pi_0 = \pi_0\left(1 + (\lambda/\mu) + (\lambda^2/2\mu^2)\right)$$
$$= 1.$$

Thus, $\pi_0 = \left(1 + (\lambda/\mu) + (\lambda^2/2\mu^2)\right)^{-1}$.

b) To accomodate the new policy, we need to alter the generator. Specifically, when both machines are working, only one operates and is thus subject to failure. The modified generator is given as:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}.$$

Following the same approach as in a) and using shorthand notation, we get

$$\pi_0 = \left(\sum_{k=0}^{2} \theta_k\right)^{-1}, \quad \theta_k = \left(\prod_{i=1}^{k} (\lambda_{i-1}/\mu_i)\right), \quad k \geq 1$$

$$\theta_0 = 1.$$

Hence,

$$\pi_0 = \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1}{\mu_2}\right)^{-1} = \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right).$$

Before we had that $\theta_2 = 1 \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{2\mu}$,

but now it is $\theta_2 = 1 \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu}$.