Problem 6.2.6

Let T be the time to extinction in the linear death process with $X_0 = N$ and α.

a) Note that $T = W_N$ and consequently

$$\mathbb{E}[T] = \mathbb{E}[W_N] = \mathbb{E}[\textstyle\sum_{i=1}^N S_i] = \sum_{i=1}^N \mathbb{E}[S_i],$$

where S_i are the independent sojourn times. In the linear death process, the death parameters are given by $\mu_k = k\alpha$, cf. p. 287.

Hence,

$$\mathbb{E}[T] = \textstyle\sum_{i=1}^N \mathbb{E}[S_i] = \sum_{i=1}^N \mu_i^{-1} = \sum_{i=1}^N (k_i\alpha)^{-1}$$

$$= \alpha^{-1} \textstyle\sum_{i=1}^N i^{-1}.$$

b) We begin as proposed in the book

$$\mathbb{E}[T] = \int_0^\infty \mathbb{P}(T > t)\,dt = \int_0^\infty (1 - F_T(t))\,dt.$$

From eq. (6.15), we know $F_T(t)$ for the linear death process.

$$\mathbb{E}[T] = \int_0^\infty \left(1 - [1 - e^{-\alpha t}]^N\right) dt.$$

Now, we use the proposed substitution $y = 1 - e^{-\alpha t}$. Then $dy/dt = \alpha e^{-\alpha t}$ and we get that $dt = (\alpha e^{-\alpha t})^{-1} dy = (\alpha(1-y))^{-1} dy$.

Hence,

$$\mathbb{E}[T] = \int_0^\infty 1-(1-e^{-\alpha t})^N dt = \int_0^1 (1-y^N)(\alpha(1-y))^{-1} dy.$$

Note the change in the integration limits, as $y(t) \to 0$ for $t \to 0$ and $y(t) \to 1$ for $t \to \infty$.

Recall that the first N terms of a geometric series is given by:

$$\sum_{k=0}^N cr^k = c(1-r^{N+1})/(1-r), \quad r \neq 1.$$

Thus,

$$\begin{aligned}
\mathbb{E}[T] &= \int_0^1 (1-y^N)(\alpha(1-y))^{-1} dy \\
&= \int_0^1 \frac{1}{\alpha} \sum_{i=0}^{N-1} y^i \, dy \\
&= \frac{1}{\alpha} \sum_{i=0}^{N-1} \int_0^1 y^i \, dy \\
&= \frac{1}{\alpha} \sum_{i=0}^{N-1} \left[\frac{1}{i+1} y^{i+1}\right]_0^1 \\
&= \frac{1}{\alpha} \sum_{i=0}^{N-1} \frac{1}{i+1} \\
&= \frac{1}{\alpha} \sum_{i=1}^{N} \frac{1}{i} = \alpha^{-1} \sum_{i=1}^{N} i^{-1}.
\end{aligned}$$