Problem 6.3.3

We consider a two-state CTMC with transition probabilities: (Note that $\tau = \alpha + \beta$)

$$P_{00}(t) = (1-\pi) + \pi e^{-\tau t}, \quad P_{01}(t) = \pi - \pi e^{-\tau t}$$
$$P_{10}(t) = (1-\pi) - (1-\pi) e^{-\tau t}, \quad P_{11}(t) = \pi + (1-\pi) e^{-\tau t}.$$

We further assume an initial distribution $(1-\pi, \pi)$. From p. 303, we know that this coincides with the limiting distribution. Therefore, we can conclude that $\mathbb{P}(V_t = 1) = \pi$ for $t \geq 0$ and $\mathbb{P}(V_t = 0) = 1 - \pi$ for $t \geq 0$. Consequently, the mean process is constant, i.e.

$$\mathbb{E}[V_t] = 1 \cdot \mathbb{P}(V_t = 1) + 0 \cdot \mathbb{P}(V_t = 0) = \pi, \quad t \geq 0.$$

This allows us to calculate $\mathbb{E}[V_s V_t]$:

$$\begin{aligned}
\mathbb{E}[V_s V_t] &= \mathbb{E}[V_s V_t \mid V_s = 0]\,\mathbb{P}(V_s = 0) \\
&\quad + \mathbb{E}[V_s V_t \mid V_s = 1]\,\mathbb{P}(V_s = 1) \\
&= \mathbb{E}[V_t \mid V_s = 1]\,\mathbb{P}(V_s = 1) \\
&= \mathbb{E}[1 \mid V_s = 1, V_t = 1]\,\mathbb{P}(V_t = 1 \mid V_s = 1)\,\mathbb{P}(V_s = 1) \\
&= 1 \cdot P_{11}(t-s) \cdot \pi \\
&= (1 - P_{10}(t-s))\,\pi \\
&= \pi - \pi P_{10}(t-s).
\end{aligned}$$

Hence,

$$\mathrm{Cov}(V_s, V_t) = \mathbb{E}[V_s V_t] - \mathbb{E}[V_s]\,\mathbb{E}[V_t]$$
$$= \pi - \pi P_{10}(t-s) - \pi^2 = \pi - \pi\left((1-\pi) - (1-\pi) e^{-\tau(t-s)}\right) - \pi^2 = \pi \cdot (1-\pi) \cdot e^{-\tau(t-s)}.$$