Problem 6.1.3

We consider the state with $0 \le K \le N$ infected individuals and $N-K$ susceptible individuals.

Each infected individual can transmit the disease to any of the $N-K$ susceptible individuals. Hence, for a time interval of length h, the probability that exactly one susceptible individual is infected by a single infected individual is given by a binomial form:

$$\binom{N-K}{1}(\alpha h + o(h))(1-\alpha h + o(h))^{N-K-1}$$

$$= (N-K)\alpha h + o_{N,K}(h).$$

For the same time interval, the probability that exactly one susceptible is infected from K infected individual is also given by a binomial probability:

$$\binom{K}{1}((N-K)\alpha h + o_{N,K}(h))(1-(N-K)\alpha h + o_{N,K}(h))^{K-1}$$

$$= K(N-K)\alpha h + o_{N,K}(h).$$

In conclusion

$$\lambda_K = \lim_{h\to 0}\left(\frac{\mathbb{P}(X_{t+h} = K+1 \mid X_t = K)}{h}\right) = \lim_{h\to 0}\left(\frac{K(N-K)\alpha h + o_{N,K}(h)}{h}\right)$$

$$= K(N-K)\alpha + \lim_{h\to 0}\left(o_{N,K}(h)/h\right) = K(N-K)\alpha.$$