Problem 6.3.1

We consider a DTMC $\{\mathcal{E}_n\}_{n \in \mathbb{N}_0}$ governed by P and a Poisson process $\{N_t\}_{t \geq 0}$ with rate λ. We then construct the process $\{X_t\}_{t \geq 0}$ as $X_t = \mathcal{E}_{N_t}$.

Since $\{X_t\}$ has the same state space as $\{\mathcal{E}_n\}$, the state space is $E = \{0, 1\}$.

Now we check the postulates in sec. 6.3.1. Postulates 4 and 5 are trivially satisfied. Next, we turn our attention to the case when $X_t = 0$. The process will jump to state one when there is an arrival in $\{N_t\}$, which happens at rate λ. (as $P_{00} = 0$ and $P_{01} = 1$).
Therefore $\lambda_0 = \lambda$.
When $X_t = 1$, the case is not as simple. Here, we need to carry out the actual calculations.

$$\mu_1 = \lim_{h \to 0} \mathbb{P}(X_{t+h} = 0 \mid X_t = 1)/h.$$

In the limit, the probability that $X_{t+h} = 0$ given $X_t = 1$ is the probability that $\{N_t\}$ has exactly one arrival and $\{\mathcal{E}_n\}$ transitions from 1 to 0. We know this as the probability of multiple arrivals in $\{N_t\}$

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in a time interval of length h will tend to zero as h tends to zero. Hence

$$\mu_1 = \lim_{h \to 0} \mathbb{P}(X_{t+h} = 0 \mid X_t = 1)/h$$

$$= \lim_{h \to 0} \mathbb{P}(N_{t+h} - N_t = 1, \varepsilon_{N_{t+h}} = 0 \mid \varepsilon_{N_t} = 1)/h$$

$$= \lim_{h \to 0} \frac{\mathbb{P}(\varepsilon_{N_{t+h}} = 0 \mid \varepsilon_{N_t} = 1, N_{t+h} - N_t = 1)\,\mathbb{P}(N_{t+h} - N_t = 1 \mid \varepsilon_{N_t} = 1)}{h}$$

Given $\varepsilon_{N_t} = 1$ and we know that a transition will happen in $\{\varepsilon_n\}$ in the interval $(t, t+h]$, the probability that $\varepsilon_{N_{t+h}} = 0$ must be $P_{10} = 1-\alpha$. Furthermore, the probability of an arrival in $\{N_t\}$ in a interval of length h, is independent of the state of $\{\varepsilon_n\}$. Hence

$$\mathbb{P}(N_{t+h} - N_t = 1 \mid \varepsilon_{N_t} = 1) = \mathbb{P}(N_{t+h} - N_t = 1)$$
$$= \lambda h + o(h).$$

In conclusion,

$$\mu_1 = \lim_{h \to 0} (1-\alpha)(\lambda h + o(h))/h$$

$$= \lim_{h \to 0} \left((1-\alpha)\lambda + (1-\alpha)o(h)/h\right)$$

$$= (1-\alpha)\lambda.$$