Exercise 6.1.2

We consider a pure birth process $\{X_t\}_{t\geq 0}$ with $X_0 = 0$, $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 5$.

Let $W_i = \inf\{t \geq 0 : X_t = i\}$, i.e. W_i is the first time $\{X_t\}_{t\geq 0}$ reaches state i.

a) Then:

$$W_3 = (W_3 - W_2) + (W_2 - W_1) + (W_1 - W_0),$$

where of course $W_0 = 0$. Each of the above terms represent a sojourn time. The first term is the sojourn time in state 0, the second term is the sojourn time in state 1, and the third term is the sojourn time in state 2. Thus,

$$\mathbb{E}[W_3] = \mathbb{E}[W_3 - W_2] + \mathbb{E}[W_2 - W_1] + \mathbb{E}[W_1 - W_0]$$

$$= \lambda_2^{-1} + \lambda_1^{-1} + \lambda_0^{-1} = 1/2 + 1/3 + 1 = 11/6.$$

b) Similarly, we get:

$$\mathbb{E}[W_2] = \mathbb{E}[W_2 - W_1] + \mathbb{E}[W_1 - W_0]$$

$$= \lambda_1^{-1} + \lambda_0^{-1} = 1/3 + 1 = 8/6,$$

$$\mathbb{E}[W_1] = \mathbb{E}[W_1 - W_0] = \lambda_0^{-1} = 1 = 6/6.$$

In conclusion:

$$\mathbb{E}[W_3 + W_2 + W_1] = \mathbb{E}[W_3] + \mathbb{E}[W_2] + \mathbb{E}[W_1] = \frac{25}{6}$$

c) We find the variance of W_3 by direct calculation:

$$\mathbb{V}[W_3] = \mathbb{V}[(W_3 - W_2) + (W_2 - W_1) + (W_1 - W_0)].$$

Since the increments are independent by definition, we get:

$$\begin{aligned}\mathbb{V}[W_3] &= \mathbb{V}[W_3 - W_2] + \mathbb{V}[W_2 - W_1] + \mathbb{V}[W_1 - W_0] \\ &= (\lambda_2^{-1})^2 + (\lambda_1^{-1})^2 + (\lambda_0^{-1})^2 \\ &= (1/2)^2 + (1/3)^2 + 1^2 = 49/36.\end{aligned}$$