Problem 6.2.1

We consider a pure death process $\{X_t\}_{t\geq 0}$ with $X_0 = N$ and death parameters $\mu_1, \mu_2, \dots, \mu_N$. Let T be an independent exponentially distributed random variable with parameter θ.

If $X_T = 0$ it means that state 0 has been reached at at most time T. So if W_N is the waiting time to reach state 0 (N transitions downwards / N deaths), we can use that $\{X_T = 0\} = \{W_N \leq T\}$. From previous lectures, we know that $W_N = \sum_{i=1}^{N} S_i$, where S_i are the independent sojourn times.

Now we calculate the probability in question:

$$\mathbb{P}(X_T = 0) = \mathbb{P}(W_N \leq T) = \mathbb{P}(\textstyle\sum_{i=1}^{N} S_i \leq T)$$

$$= \mathbb{P}(\textstyle\sum_{i=1}^{N} S_i \leq T \mid \sum_{i=1}^{N-1} S_i \leq T)\,\mathbb{P}(\sum_{i=1}^{N-1} S_i \leq T).$$

The memoryless property of T implies that the conditional probability above simplifies to $\mathbb{P}(S_N \leq T)$. We know that this probability is given by $\mu_N / (\mu_N + \theta)$ from earlier exercises (alternatively, see p. 297).

In summary,

$$\mathbb{P}(X_T = 0) = \mathbb{P}(\sum_{i=1}^{N} S_i \leq T) = \mu_N (\mu_N + \theta)^{-1} \mathbb{P}(\sum_{i=1}^{N-1} S_i \leq T).$$

Using the last equality repeatedly, we get that:

$$\mathbb{P}(X_T = 0) = \prod_{i=1}^{N} \mu_i / (\mu_i + \theta).$$