Problem 6.1.4

Let $\{N_t\}_{t\geq 0}$ denote the arrival process for customers acquired through media advertising and let $\{M_t\}_{t\geq 0}$ denote the arrival process for customers acquired through WoM advertising.

Then $\{N_t\}$ and $\{M_t\}$ are independent Poisson processes. (We can safely assume that $\{M_t\}$ is a Poisson process as we are only given a mean rate). The rate of $\{N_t\}$ is $\alpha = 1$ customer per month. The rate of $\{M_t\}$ is dependent on current number of customers, specifically $\beta(t) = \theta X(t) = 2X(t)$ customers per month.

a) Since the sum of independent Poisson processes is a Poisson process with rate equal to ~~sum~~ the sum of the rates.

Thus, $\{X_t\}_{t\geq 0}$ is a Poisson processes given by $X_t = N_t + M_t$ with rate $\lambda(t) = \alpha + \theta X_t$, which translates into the birth parameters $\lambda_K = \alpha + \theta K = 1 + 2K$.

b) To find the probability that exactly two items have been sold during the first month, we shall apply formulas (6.8) and (6.9). We can do this as none of the birth parameters are equal. From eq. (6.9):

$$B_{0,2} = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8},$$
$$B_{1,2} = \left(-\frac{1}{2}\right)\cdot\frac{1}{2} = -\frac{1}{4},$$
$$B_{2,2} = \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{1}{8}.$$

From eq. (6.8) we conclude that

$$\mathbb{P}(X_1 = 2 \mid X_0 = 0) = \lambda_0\lambda_1\left(B_{0,2}e^{-\lambda_0} + B_{1,2}e^{-\lambda_1} + B_{2,2}e^{-\lambda_2}\right)$$
$$= 1\cdot 3\cdot\left(\frac{1}{8}e^{-1} - \frac{1}{4}e^{-3} + \frac{1}{8}e^{-5}\right)$$
$$= 0.103.$$