Problem 6.4.2

We consider a birth and death process with parameters $\lambda_n = \lambda$ for $n \in \mathbb{N}_0$ and $\mu_n = \mu$ for $n \in \mathbb{N}$.

According to eq. (6.37) the stationary distribution exists whenever the sum $\sum_{i=0}^{\infty} \theta_i = \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i$ converges. Therefore, the convergence criterion reduces to $\lambda < \mu$.

The fraction λ/μ is often refered to as the traffic intensity in queueing theory and denoted by ρ. Thus, our convergence criterion is $\rho < 1$. Whenever this is satisfied, eq. (6.37) gives the stationary distribution as

$$\pi_i = \theta_i / (\textstyle\sum_{k=0}^{\infty} \theta_k).$$

Since the sum converges, we know that

$$\sum_{k=0}^{\infty} \theta_k = \sum_{k=0}^{\infty} \rho^k = (1-\rho)^{-1} = (1-\lambda/\mu)^{-1}.$$

Thus,

$$\pi_i = \theta_i / (\textstyle\sum_{k=0}^{\infty} \theta_k) = \rho^i / (1-\rho)^{-1} = \rho^i (1-\rho)$$

$$= (\lambda/\mu)^i (1 - \lambda/\mu).$$