Exercise 6.2.2

We consider a pure death process $\{X_t\}_{t\geq 0}$ with $X_0 = 3$, $\mu_0 = 0$, $\mu_1 = 3$, $\mu_2 = 2$, and $\mu_3 = 5$.

Let $\omega_i = \inf\{t \geq 0 : X_t = 3 - i\}$, i.e. ω_i is the first time the process reaches state $3-i$.

a) Then,

$$\omega_3 = (\omega_3 - \omega_2) + (\omega_2 - \omega_1) + (\omega_1 - \omega_0),$$

where of course $\omega_0 = 0$ and the increments are independent. We have that $\omega_i - \omega_{i-1}$ follows an exponential(μ_{4-i}) distribution.

Hence,

$$\mathbb{E}[\omega_3] = \mathbb{E}[\omega_3 - \omega_2] + \mathbb{E}[\omega_2 - \omega_1] + \mathbb{E}[\omega_1 - \omega_0]$$

$$= \mu_1^{-1} + \mu_2^{-1} + \mu_3^{-1} = 1/3 + 1/2 + 1/5 = \frac{31}{30}.$$

Similarly,

b) $$\mathbb{E}[\omega_3 + \omega_2 + \omega_1] = \mathbb{E}[\omega_3] + \mathbb{E}[\omega_2] + \mathbb{E}[\omega_1]$$

$$= (\mu_1^{-1} + \mu_2^{-1} + \mu_3^{-1}) + (\mu_2^{-1} + \mu_3^{-1}) + \mu_3^{-1}$$

$$= 31/30 + 21/30 + 6/30 = 58/30 = 29/15.$$

c) Finally,

$$\mathbb{V}[\omega_3] = \mathbb{V}[\omega_3 - \omega_2] + \mathbb{V}[\omega_2 - \omega_1] + \mathbb{V}[\omega_1 - \omega_0]$$

$$= (\mu_1^{-1})^2 + (\mu_2^{-1})^2 + (\mu_3^{-1})^2 = 361/900.$$