This problem essentially asks you to show that a Binomial distribution converges (in this course, we shall not discuss the specific type of convergence) to a Poisson distribution when the number of trials tends to infinity and the success parameter tends to zero. This is usually refered to as "The law of rare events" or "The Poisson limit theorem".

From the problem, we see that $X_i \sim \text{Bin}(N, \frac{1}{M})$ or similarly $X_i \sim \text{Bin}(N, \lambda/N)$. Thus,

$$\mathbb{P}(X_i = k) = \binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k}$$

$$= \frac{N!}{k!(N-k)!}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^N\left(1-\frac{\lambda}{N}\right)^{-k}.$$

We rearrange the terms and use that $1-\frac{\lambda}{N} = \frac{N-\lambda}{N}$.

$$\mathbb{P}(X_i = k) = \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^N\frac{N!}{(N-k)!}\left(\frac{1}{N}\right)^k\left(\frac{N}{N-\lambda}\right)^k$$

$$= \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^N\frac{N!}{(N-k)!}\left(\frac{1}{N-\lambda}\right)^k.$$

Recall that $N! = N\cdot(N-1)\cdot\ldots\cdot(N-(k-1))\cdot(N-k)!$.
Note that there are k factors besides $(N-k)!$.
Thus,

$$\mathbb{P}(X_i = k) = \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{N}\right)^N\frac{N}{N-\lambda}\cdot\frac{N-1}{N-\lambda}\cdot\ldots\cdot\frac{N-(k-1)}{N-\lambda}.$$

For $N \to \infty$, the last k factors tend to one and consequently

$$\mathbb{P}(X_i = k) \to \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } N \to \infty.$$

Next, to show independence between X_i and X_j for $i \neq j$, we show that their joint law is the product of the marginal laws.

The joint distribution of X_i and X_j is a multinomial distribution with parameters $(N, \frac{1}{M}, \frac{1}{M})$ or similarly $(N, \frac{\lambda}{N}, \frac{\lambda}{N})$. Hence,

$$\mathbb{P}(X_i = k, X_j = l) = \frac{N!}{k!\,l!\,(N-k-l)!} \left(\frac{\lambda}{N}\right)^k \left(\frac{\lambda}{N}\right)^l \left(1 - 2\frac{\lambda}{N}\right)^{N-k-l}.$$

Using similar arguments as before, we get

$$\mathbb{P}(X_i = k, X_j = l) = \left(\frac{\lambda^k}{k!}\right)\left(\frac{\lambda^l}{l!}\right)\left(1 - \frac{2\lambda}{N}\right)^N \frac{N!}{(N-k-l)!} \left(\frac{1}{N}\right)^{k+l} \left(\frac{N-2\lambda}{N}\right)^{-(k+l)}$$

$$= \left(\frac{\lambda^k}{k!}\right)\left(\frac{\lambda^l}{l!}\right)\left(1 - \frac{2\lambda}{N}\right)^N \frac{N!}{(N-k-l)!} \left(\frac{1}{N-2\lambda}\right)^{k+l}.$$

For $N \to \infty$, we get

$$\mathbb{P}(X_i = k, X_j = l) = \left(\frac{\lambda^k}{k!}\right)\left(\frac{\lambda^l}{l!}\right) e^{-2\lambda} = \left(\frac{\lambda^k}{k!} e^{-\lambda}\right)\left(\frac{\lambda^l}{l!} e^{-\lambda}\right).$$

In conclusion,

$$\lim_{N \to \infty} \mathbb{P}(X_i = k, X_j = l) = \lim_{N \to \infty} \mathbb{P}(X_i = k) \cdot \lim_{N \to \infty} \mathbb{P}(X_j = l)$$

$$= \lim_{N \to \infty} \mathbb{P}(X_i = k)\, \mathbb{P}(X_j = l).$$

Problem 5.2.6

Finally, the fraction of storage locations that have two or more accounts assigned to them is given as

$$\mathbb{P}(X_i \geq 2) = 1 - \mathbb{P}(X_i = 0) - \mathbb{P}(X_i = 1).$$

Hence,

$$\lim_{N \to \infty} \mathbb{P}(X_i \geq 2) = 1 - \lim_{N \to \infty} \mathbb{P}(X_i = 0) - \lim_{N \to \infty} \mathbb{P}(X_i = 1)$$

$$= 1 - e^{-\lambda} - \lambda e^{-\lambda}.$$