Problem 5.1.9

In this problem $\{X_t\}_{t\geq 0}$ is a Poisson process with rate $\lambda = 2$ per time unit.

a) Since $X_0 = 0$, we have that

$$X_t - X_0 = X_t \sim \text{Pois}(\lambda t).$$

Therefore, $\mathbb{E}[X_t] = \lambda t$.

Moreover, it follows that $\mathbb{V}[X_t] = \mathbb{E}[X_t] = \lambda t$.

From the computational formula for variance, we can conclude that

$$\mathbb{E}[X_t^2] = \mathbb{V}[X_t] + \mathbb{E}[X_t]^2 = \lambda t + (\lambda t)^2.$$

b) Now, we can find the unconditional values as.

$$\mathbb{E}[X_T] = \int_0^1 \mathbb{E}[X_T \mid T = t]\, f_T(t)\, dt$$

$$= \int_0^1 \lambda t \cdot 1 \, dt = \lambda \left[\tfrac{1}{2} t^2\right]_0^1 = \tfrac{1}{2}\lambda = 1$$

as $\lambda = 2$. Similarly we get that

$$\mathbb{E}[X_T^2] = \int_0^1 \mathbb{E}[X_T^2 \mid T = t]\, f_T(t)\, dt$$

$$= \int_0^1 \lambda t + (\lambda t)^2 \, dt = \tfrac{1}{2}\lambda + \tfrac{1}{3}\lambda^2 = 1 + \tfrac{4}{3}.$$

Hence, $\mathbb{V}[X_T] = \mathbb{E}[X_T^2] - \mathbb{E}[X_T]^2 = 4/3.$