Exercise 5.4.1

In this exercise, $\{X_t\}_{t\geq 0}$ is a Poisson process with rate λ.

The exercise can be solved easily using the Beta distribution, however we shall present a solution using the material from Ch. 5.

From Theorem 5.7, we have the conditional joint density

$$f_{\underline{W}}(\underline{w} \mid X_t = n) = n!\, t^{-n} \text{ for } D = \{\underline{w} : 0 < w_1 < \cdots < w_n \leq t\},$$

where $\underline{W} = (W_1, \ldots, W_n)$ and $\underline{w} = (w_1, \ldots, w_n)$.

We then apply the formula

$$\mathbb{E}[W_1 \mid X_1 = n] = \int_D w_1 \cdot f_{\underline{W}}(\underline{w} \mid X_t = n)\, dw_1 \ldots dw_n.$$

$$= \int_0^1 \int_0^{w_n} \int_0^{w_{n-1}} \cdots \int_0^{w_2} w_1 \cdot n!\, dw_1 \ldots dw_n$$

$$= (n+1)^{-1}.$$

To evaluate the integral, we use induction to show that

$$\int_0^{w_n} \cdots \int_0^{w_2} w_1\, dw_1 \cdots dw_{n-1} = (n!)^{-1} w_n^n.$$

The base case $n = 2$ is easy to show.

Exercise 5.4.1

For the induction step, we use weak induction and assume it is true for some $n \geq 2$. Then

$$\int_0^{w_{n+1}} \cdots \int_0^{w_2} w_1 \, dw_1 \cdots dw_n = \int_0^{w_{n+1}} \left(\int_0^{w_n} \cdots \int_0^{w_2} w_1 \, dw_1 \cdots dw_{n-1} \right) dw_n$$

$$= \int_0^{w_{n+1}} (n!)^{-1} w_n^n \, dw_n$$

$$= [(n+1)!]^{-1} w_{n+1}^{n+1}.$$

Hence, the induction hypothesis is true and the result follows directly.

An alternative approach is the following.

Let $W_1^{(n)}$ be the first of the arrivals conditioned on $X_1 = n$. Then we know that $W_1^{(n)} \overset{d}{=} U_{(1)}^{(n)}$, where $U_{(1)}^{(n)} = \min(U_1, \ldots, U_n)$, where the $U_i \sim \mathcal{U}(0,1)$ and are independent. Then

$$F_{U_{(1)}^{(n)}}(u) = \mathbb{P}(U_{(1)}^{(n)} \leq u) = 1 - (1-u)^n, \text{ and}$$

$$f_{U_{(1)}^{(n)}}(u) = \frac{d}{du} F_{U_{(1)}^{(n)}}(u) = n(1-u)^{n-1}, \text{ for } u \in [0,1].$$

This leads to the expectation

$$\mathbb{E}[U_{(1)}^{(n)}] = \int_0^1 u \cdot f_{U_{(1)}^{(n)}}(u) \, du = (n+1)^{-1}.$$