Problem 5.4.8

This problem is solved by invoking the uniformity-theory from sec 5.4.

$$\mathbb{E}[Z_t] = \mathbb{E}\left[\sum_{k=1}^{N(t)} \theta_k(t)\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{N(t)} \varepsilon_k e^{-\alpha(t-w_k)}\right].$$

Now, we invoke the law of total expectation

$$\mathbb{E}[Z_t] = \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, \mathbb{E}\left[\sum_{k=1}^{N(t)} \varepsilon_k e^{-\alpha(t-w_k)} \,\Big|\, N(t)=n\right]$$

Invoking Theorem 5.7 yields

$$\mathbb{E}[Z_t] = \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, \mathbb{E}\left[\sum_{k=1}^{n} \varepsilon_k e^{-\alpha(t-U_k)}\right].$$

You should confer with p. 253 + 254 to see why this works. Essentially, Th. 5.7 states that $(w_1, \dots, w_n)$ has the same distribution as $(U_{(1)}, \dots, U_{(n)})$ for $U_1, \dots, U_n$ are i.i.d with $U_i \sim U(0,1)$ when $X_1 = n$. So

$$\sum_{i=1}^{n} w_i \overset{d}{=} \sum_{i=1}^{n} U_{(i)} \overset{E}{=} \sum_{i=1}^{n} U_i \quad \text{for } U_i \sim U(0,1) \text{ i.i.d}$$

when we condition on $X_1 = n$. Similarly

$$\sum_{i=1}^{n} h(w_i) \overset{d}{=} \sum_{i=1}^{n} h(U_{(i)}) \overset{E}{=} \sum_{i=1}^{n} h(U_i)$$

under the same conditions. Here "$\overset{d}{=}$" means equal in distribution and "$\overset{E}{=}$" means equal in expectation.

We continue with $U_k \sim U(0,t)$ i.i.d.

$$\mathbb{E}[Z_t] = \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, \mathbb{E}\left[\sum_{k=1}^{n} \varepsilon_k e^{-\alpha(t-U_k)}\right]$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, \mathbb{E}\left[\sum_{k=1}^{n} \varepsilon_k e^{-\alpha U_k}\right]$$

as $U_k \overset{d}{=} (t-U_k)$. by symetry. We then invoke the independence of ε_k and U_k. Thus,

$$\mathbb{E}[Z_t] = \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n) \sum_{k=1}^{n} \mathbb{E}[\varepsilon_k]\, \mathbb{E}[e^{-\alpha U_k}],$$

where we also used the linearity of the expectation operator. We let $\mathbb{E}[\varepsilon_k] = \mu_\varepsilon$. Furthermore,

$$\mathbb{E}[e^{-\alpha U_k}] = \int_0^t e^{-\alpha u} t^{-1}\, du \quad (\text{Expectation of function})$$

$$= (\alpha t)^{-1}(1-e^{-\alpha t}).$$

Hence

$$\mathbb{E}[Z_t] = \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, n\, \mu_\varepsilon (\alpha t)^{-1}(1-e^{\alpha t})$$

$$= \mu_\varepsilon (\alpha t)^{-1}(1-e^{-\alpha t}) \sum_{n=1}^{\infty} \mathbb{P}(N(t)=n)\, n$$

$$= \mu_\varepsilon (\alpha t)^{-1}(1-e^{-\alpha t})\, \mathbb{E}[N(t)]$$

As $\mathbb{E}[N(t)] = \lambda t$, we get:

$$\mathbb{E}[Z_t] = \mu_\varepsilon \lambda \alpha^{-1}(1-e^{-\alpha t}).$$