Problem 5.3.9

Let $\{X_t^1\}_{t\geq 0}$ and $\{X_t^2\}_{t\geq 0}$ be independent Poisson processes with rates λ_1 and λ_2, respectively.

a) Let ω_1^1 and ω_1^2 denote the first arrival times for $\{X_t^1\}$ and $\{X_t^2\}$, respectively.
Then $\omega_1^1 \sim \exp(\lambda_1)$ and $\omega_1^2 \sim \exp(\lambda_2)$ independently. Thus,

$$
\begin{aligned}
\mathbb{P}(\omega_1^{(1)} < \omega_1^{(2)}) &= \int_0^\infty \mathbb{P}(\omega_1^{(1)} < \omega_1^{(2)} \mid \omega_1^{(2)} = \omega)\, f_{\omega_1^{(2)}}(\omega)\, d\omega \\
&= \int_0^\infty \mathbb{P}(\omega_1^{(1)} < \omega)\, f_{\omega_1^{(2)}}(\omega)\, d\omega \\
&= \int_0^\infty (1 - e^{-\lambda_1 \omega})\lambda_2 e^{-\lambda_2 \omega}\, d\omega \\
&= \int_0^\infty \lambda_2 e^{-\lambda_2 \omega}\, d\omega - \int_0^\infty e^{-(\lambda_1+\lambda_2)\omega}\lambda_2\, d\omega \\
&= 1 - \lambda_2/(\lambda_1+\lambda_2) \\
&= \lambda_1/(\lambda_1+\lambda_2).
\end{aligned}
$$

You should remember this key result as it is important for CTMCs.

b)

Problem 5.3.9

b) To find the result for part b, we only need few new calculations. We shall use sojourn times as from Theorem 5.5. We have $W_1^{(1)} = S_1^{(1)}$, $W_1^{(2)} = S_1^{(2)}$, $W_2^{(1)} = S_1^{(1)} + S_2^{(1)}$, and $W_2^{(2)} = S_1^{(2)} + S_2^{(2)}$. All the sojourn times are independent and exponentially distributed.

We also need to show "the memoryless property with random variables". That is

$$\mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} \mid S_1^{(1)} < S_1^{(2)}) = \mathbb{P}(S_2^{(1)} < S_1^{(2)}).$$

This can be shown as

$$\mathbb{P}(S_1^{(1)} + S_{2}^{(1)} < S_1^{(2)} \mid S_1^{(1)} < S_1^{(2)})$$

$$= \int_0^\infty \mathbb{P}(t + S_{2}^{(1)} < S_1^{(2)} \mid t < S_1^{(2)}) f_{S_1^{(1)}}(t)\, dt$$

$$= \int_0^\infty \int_0^\infty \mathbb{P}(t + s < S_1^{(2)} \mid t < S_1^{(2)}) f_{S_1^{(1)}}(t) f_{S_2^{(1)}}(s)\, dt\, ds$$

$$= \int_0^\infty \int_0^\infty \mathbb{P}(s < S_1^{(2)}) f_{S_1^{(1)}}(t) f_{S_2^{(1)}}(s)\, dt\, ds$$

$$= \int_0^\infty \mathbb{P}(s < S_1^{(2)}) f_{S_2^{(1)}}(s) \int_0^\infty f_{S_1^{(1)}}(t)\, dt\, ds$$

$$= \int_0^\infty \mathbb{P}(s < S_1^{(2)}) f_{S_2^{(1)}}(s)\, ds = \mathbb{P}(S_2^{(1)} < S_1^{(2)}).$$

Using these results, we get the result simply as:

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$$
\begin{aligned}
\mathbb{P}(\omega_2^{(1)} < \omega_2^{(2)}) &= \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} + S_2^{(2)}) \\
&= \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} + S_2^{(2)} \mid S_1^{(1)} + S_2^{(1)} < S_1^{(2)},\ S_1^{(1)} < S_1^{(2)}) \\
&\quad \cdot \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} \mid S_1^{(1)} < S_1^{(2)})\, \mathbb{P}(S_1^{(1)} < S_1^{(2)}) \\
&\quad + \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} + S_2^{(2)} \mid S_1^{(1)} + S_2^{(1)} > S_1^{(2)},\ S_1^{(1)} < S_1^{(2)}) \\
&\quad \cdot \mathbb{P}(S_1^{(1)} + S_2^{(1)} > S_1^{(2)} \mid S_1^{(1)} < S_1^{(2)})\, \mathbb{P}(S_1^{(1)} < S_1^{(2)}) \\
&\quad + \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_1^{(2)} + S_2^{(2)} \mid S_1^{(1)} + S_2^{(1)} > S_1^{(2)},\ S_1^{(1)} > S_1^{(2)}) \\
&\quad \cdot \mathbb{P}(S_1^{(1)} + S_2^{(1)} > S_1^{(2)} \mid S_1^{(1)} > S_1^{(2)})\, \mathbb{P}(S_1^{(1)} > S_1^{(2)}) \\
&= 1 \cdot \mathbb{P}(S_2^{(1)} < S_1^{(2)}) \cdot \mathbb{P}(S_1^{(1)} < S_1^{(2)}) \\
&\quad + \mathbb{P}(S_2^{(1)} < S_2^{(2)}) \cdot \mathbb{P}(S_1^{(2)} < S_2^{(1)}) \cdot \mathbb{P}(S_1^{(1)} < S_1^{(2)}) \\
&\quad + \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_2^{(2)}) \cdot 1 \cdot \mathbb{P}(S_1^{(1)} > S_1^{(2)}) \\
&= (\lambda_1/(\lambda_1+\lambda_2))(\lambda_1/(\lambda_1+\lambda_2)) \\
&\quad + (\lambda_1/(\lambda_1+\lambda_2))(\lambda_2/(\lambda_1+\lambda_2))(\lambda_1/(\lambda_1+\lambda_2)) \\
&\quad + \mathbb{P}(S_1^{(1)} + S_2^{(1)} < S_2^{(2)} \mid S_1^{(1)} < S_2^{(2)})\, \mathbb{P}(S_1^{(1)} < S_2^{(2)})(\lambda_2/(\lambda_1+\lambda_2)) \\
&= (\lambda_1/(\lambda_1+\lambda_2))^2 + (\lambda_1/(\lambda_1+\lambda_2))^2(\lambda_2/(\lambda_1+\lambda_2)) \\
&\quad + \mathbb{P}(S_2^{(1)} < S_2^{(2)})\, \mathbb{P}(S_1^{(1)} < S_2^{(2)})(\lambda_2/(\lambda_1+\lambda_2)) \\
&= (\lambda_1/(\lambda_1+\lambda_2))^2 + (\lambda_1/(\lambda_1+\lambda_2))^2(\lambda_2/(\lambda_1+\lambda_2)) \\
&\quad + (\lambda_1/(\lambda_1+\lambda_2))^2(\lambda_2/(\lambda_1+\lambda_2)) \\
&= (\lambda_1/(\lambda_1+\lambda_2))^2\left(1 + 2(\lambda_2/(\lambda_1+\lambda_2))\right).
\end{aligned}
$$