Problem 5.4.1

Let W_i denote the i'th arrival time for a Poisson process $\{X_t\}_{t\geq 0}$. We know that $X_1 = n$ and $W_K = w$ for some $K < n$. The latter condition can be restated as $X_w = K$. We know that in the time interval $[0, w]$ we have K arrivals and in the time interval $(w, 1]$ we have $n-K$ arrivals.

For the first set of arrivals, Theorem 5.7 yields:

$$f_{W_1 \dots W_{K-1}}(w_1, \dots, w_{K-1}) = (K-1)!\, w^{-(K-1)}.$$

Similarly for the second set

$$f_{W_{K+1} \dots W_n}(w_{K+1}, \dots, w_n) = (n-K)!\,(1-w)^{-(n-K)}$$

since there were $(n-K)$ arrivals in a time interval of length $(1-w)$. As we know the location of W_K, we can separate the sets of arrivals, which are now independent. They are independent since we cannot infere information from one set by know-ing the other set. Therefore, the joint distribution is found by multiplying the above functions:

$$f_{W_1 \dots W_{K-1} W_{K+1} \dots W_n}(w_1, \dots, w_{K-1}, w_{K+1}, \dots, w_n) = (K-1)!\, w^{-(K-1)} (n-K)!\,(1-w)^{-(n-K)}.$$