Problem 5.6.3

Let $\{N_t\}_{t\geq 0}$ be a Poisson process with rate λ.
Furthermore, let $Y_i \sim \mathrm{Geo}(p)$ be i.i.d and define

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

meaning $\{X_t\}_{t\geq 0}$ is a compound Poisson process.

We define T as the time to failure and follow the derivation on p. 266-267.

$$\mathbb{E}[T] = \lambda^{-1} \sum_{n=0}^{\infty} G^{(n)}(\alpha - 1).$$ - Total damage should be strictly less than α.

From sec. 1.3.3 (p. 22), we know that $\sum_{i=1}^{n} Y_i \sim NB(n, p)$. Consequently

$$\mathbb{E}[T] = \lambda^{-1} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\alpha-1} \binom{n+k-1}{k} p^n (1-p)^k \right)$$

We will separate the first term of the outer sum from the rest and change the order of summations:

$$\mathbb{E}[T] = \lambda^{-1} \left(1 + \sum_{k=0}^{\alpha-1} p(1-p)^k \sum_{n=1}^{\infty} \binom{n+k-1}{k} p^{n-1} \right).$$

Since binomial coefficients are symmetric, we have $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$. Then we apply eq. (1.71) on p. 45 to get that

$$\sum_{n=1}^{\infty} \binom{n+k-1}{k} p^{n-1} = \sum_{n=1}^{\infty} \binom{k+(n-1)}{n-1} p^{n-1} = \sum_{n=0}^{\infty} \binom{(k+1)+n-1}{n} p^{n-1}$$

$$= (1-p)^{-(k+1)}.$$

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Hence,

$$\mathbb{E}[T] = \lambda^{-1}\left(1 + \sum_{k=0}^{\alpha-1} p(1-p)^{k}(1-p)^{-(k+1)}\right)$$

$$= \lambda^{-1}\left(1 + \sum_{k=0}^{\alpha-1} p/(1-p)\right)$$

$$= \lambda^{-1}(1 + \alpha p/(1-p)).$$