Problem 4.4.7 (Long)

We shall give a solution, which is not the most condensed version of the solution, but the most elaborate version.

In this version, we interpret the problem such that it rains in any given period with probability p independent of the current and past states of the weather.

We introduce the DTMC $\{X_n\}_{n \in \mathbb{N}_0}$, whose statespace is defined by 3 variables:

- The Location of the man: Home (H) or Office (O)
- The weather: Rain (R) or sunny (S)
- The Location of the car: Home (CH) or office (CO).

Each state is the composed of a unique combination of the above variables. The transition probability matrix associated with $\{X_n\}$ is then given as

$$\begin{array}{c} (H,S,CH) \\ (H,S,CO) \\ (H,R,CH) \\ (H,R,CO) \\ (O,S,CH) \\ (O,S,CO) \\ (O,R,CH) \\ (O,R,CO) \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-p & 0 & p \\ 1-p & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4.4.7 (Long)

From the matrix and the problem description, we know that some of the transitions are due to deterministic effects, e.g. we constantly change the location of the man. Furthermore, we note that the location of the car is irrevelant as we only need to know whether it is at the location of the individual. In summary, the state of X_n can be described by two factors:

- The weather,
- Whether the car and the individual are at the same location: YES (1) or NO (0).

We then reduce P to:

$$\begin{array}{c} (S,1) \\ (S,0) \\ (R,1) \\ (R,0) \end{array} \begin{bmatrix} 0 & 1-p & 0 & p \\ 1-p & 0 & p & 0 \\ 1-p & 0 & p & 0 \\ 1-p & 0 & p & 0 \end{bmatrix}$$

We also note that we are now interested in $\pi_{(R,0)}$, which is found by solving the system:

$$\pi_{(S,1)} = (1-p)\left(\pi_{(S,0)} + \pi_{(R,1)} + \pi_{(R,0)}\right),$$
$$\pi_{(S,0)} = (1-p)\,\pi_{(S,1)},$$
$$\pi_{(R,1)} = p\left(\pi_{(S,0)} + \pi_{(R,1)} + \pi_{(R,0)}\right),$$
$$\pi_{(R,0)} = p\,\pi_{(S,1)},$$
$$\pi_{(S,1)} + \pi_{(S,0)} + \pi_{(R,1)} + \pi_{(R,0)} = 1.$$

Problem 4.4.7 (Long)

We get the solution

$$(\pi_{(S,1)} + \pi_{(S,0)} + \pi_{(R,1)} + \pi_{(R,0)}) = ((1-p)/(2-p), (1-p)^2/(2-p), p/(2-p), p(1-p)/(2-p))$$

Thus; $\pi_{(R,0)} = p(1-p)/(2-p)$.

This is the fraction of trips where the individual will walk in the rain. The fraction of days the individual gets wet is the probability that the individual gets wet in at least one of two ~~conseq~~ consecutive trips. Thus;

$$\mathbb{P}(X_n = (R,0) \vee X_{n+1} = (R,0)).$$

We simply apply the inclusion-exclusion formula:

$$\mathbb{P}(X_n = (R,0) \vee X_{n+1} = (R,0))$$

$$= \mathbb{P}(X_n = (R,0)) + \mathbb{P}(X_{n+1} = (R,0)) - \mathbb{P}(X_n = (R,0), X_{n+1} = (R,0))$$

$$= \pi_{(R,0)} + \pi_{(R,0)} - \mathbb{P}(X_{n+1} = (R,0) \mid X_n = (R,0))\,\mathbb{P}(X_n = (R,0))$$

$$= \pi_{(R,0)} + \pi_{(R,0)} - P_{(R,0),(R,0)} \cdot \pi_{(R,0)}.$$

The last term is zero, cf. the matrix P. The idea is that the individual cannot get wet in both of two consecutive trips as the individual will be with the car at least once. Hence, the sought after probability is $2\pi_{(R,0)} = 2p(1-p)/(2-p)$.

Problem 4.4.7 (Long)

In the case with two cars, see the short solution.

Problem 4.4.7 (short)

We shall present the most concise solution to the problem here.

We consider a DTMC $\{X_n\}_{n \in \mathbb{N}_0}$, which describes the number of cars at location of the individual. The associated transition probability matrix is then given as

$$P = \begin{array}{c} 0 \\ 1 \end{array} \begin{bmatrix} 0 & 1 \\ 1-p & p \end{bmatrix}.$$

For this P, a transition refers to a trip. We check that P is regular and apply Theorem 4.1:

$$\pi_0 = (1-p)\pi_1,$$
$$\pi_1 = \pi_0 + p\pi_1,$$
$$\pi_0 + \pi_1 = 1.$$

The solution is found as $(\pi_0, \pi_1) = \left((1-p)/(2-p), 1/(2-p)\right)$. Let R denote the event that it rains and not that R is independent of the state of X_n. Thus, $\mathbb{P}(X_n = 0, R) = \mathbb{P}(X_n = 0)\mathbb{P}(R) = \pi_0 \cdot p$.

The above result is the fraction of trips, the individual has to walk in the rain.

Problem 4.4.7 (Short)

The fraction of days the individual will walk in the rain is the probability that the individual will walk in the rain at least once during two consecutive trips.

To calculate this, we apply the inclusion-exclusion formula. Let R_1 and R_2 denote the events of rain during the first and second trip, respectively. Then

$$\mathbb{P}((X_n=0, R_1) \vee (X_{n+1}=0, R_2))$$

$$= \mathbb{P}(X_n=0, R_1) + \mathbb{P}(X_{n+1}=0, R_2) - \mathbb{P}(X_n=0, X_{n+1}=0, R_1, R_2).$$

The last term is zero as $P_{00}=0$. More explicitely

$$\mathbb{P}(X_n=0, X_{n+1}=0, R_1, R_2) = \mathbb{P}(X_{n+1}=0 \mid X_n=0)\mathbb{P}(X_n=0)\mathbb{P}(R_1)\mathbb{P}(R_2)$$

$$= P_{00} \cdot \pi_0 \cdot p^2 = 0.$$

Hence, $\mathbb{P}((X_n=0, R_1) \vee (X_{n+1}=0, R_2)) = \pi_0 p + \pi_0 p = 2p\pi_0$.

For the case with two cars, we change P:

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

Problem 4.4.7 (short)

Similarly to the previous case, we find π_0 by solving the system.

$\pi_0 = (1-p)\pi_2,$
$\pi_1 = (1-p)\pi_1 + p\pi_2,$
$\pi_2 = \pi_0 + p\pi_1,$
$\pi_0 + \pi_1 + \pi_2 = 1.$

We find that $(\pi_0, \pi_1, \pi_2) = \left({}^{(1-p)}/(3-p), {}^{1}/(3-p), {}^{1}/(3-p)\right)$.

Thus, the fraction of trips, where the individual has to walk in the rain is $p \cdot \pi_0$.
The fraction of days is $2 \cdot p \cdot \pi_0 = 2p(1-p)/(3-p)$.

The two last conclusions follow from the same arguments as before with one car.