Exercise 4.4.2

We consider a DTMC $\{X_n\}_{n \in \mathbb{N}_0}$ governed by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.1 & 0.4 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 & 0 \end{bmatrix}.$$

a) To apply Theorem 4.1, we check that P is regular. Here, P is regular as

1. All states communicate, and
2. At least one state is aperiodic.

Thus, by Theorem 4.1:

$$\cancel{\pi_0 = 0.1 + \pi} \quad \begin{aligned} \pi_0 &= 0.1\pi_1 + 0.2\pi_2 + 0.3\pi_3, \\ \pi_1 &= \pi_0 + 0.4\pi_1 + 0.2\pi_2 + 0.3\pi_3, \\ \pi_2 &= 0.2\pi_1 + 0.5\pi_2 + 0.4\pi_3, \\ \pi_3 &= 0.3\pi_1 + 0.1\pi_2, \\ \pi_0 + \pi_1 &+ \pi_2 + \pi_3 = 1. \end{aligned}$$

The system has the solution

$(\pi_0, \pi_1, \pi_2, \pi_3) = (161/1111, 460/1111, 320/111, 170/1111).$

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b) To calculate m_{10}, we apply eq. (3.89). In this setting $m_{10} = v_1$. From eq. (3.89), we get:

$$v_1 = 1 + 0.4 v_1 + 0.2 v_2 + 0.3 v_3,$$
$$v_2 = 1 + 0.2 v_1 + 0.5 v_2 + 0.1 v_3,$$
$$v_3 = 1 + 0.3 v_1 + 0.4 v_2,$$

which yields the solution
$(v_1, v_2, v_3) = (950/161, 860/161, 790/161)$.

Hence, $m_{10} = v_1 = 950/161$.

c) The mean return time to state zero is given as $m_0 = 1 + m_{10} = 1111/161$.

Indeed, we therefore have that
$m_0^{-1} = 161/1111 = \pi_0$.