Problem 4.1.1

Let $\{X_n\}$ be a DTMC describing the number of balls in urn A, such that for $n \in \mathbb{N}_0$, X_n is the number of balls in urn A after n selections.

The associated transition probability matrix is given by

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

We check that P is regular: It is since

1. All states communicate, and
2. At least one state (0 and 5) is aperiodic.

Theorem 4.1 then yields the system:

$$\pi_0 = 0.5\pi_0 + 0.5\pi_1, \qquad \pi_3 = 0.5\pi_2 + 0.5\pi_4,$$
$$\pi_1 = 0.5\pi_0 + 0.5\pi_2, \qquad \pi_4 = 0.5\pi_3 + 0.5\pi_5,$$
$$\pi_2 = 0.5\pi_1 + 0.5\pi_3, \qquad \pi_5 = 0.5\pi_4 + 0.5\pi_5,$$
$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1.$$

The solution is then

$$\pi_0 = \pi_1 = \dots = \pi_5 = 1/6.$$

Problem 4.1.1

This solution is not surprising, and in fact we don't need Theorem 4.1 to reach the solution.

Note that P is doubly stochastic. According to sec. 4.1.1, this means that the limiting distribution is the uniform distribution and conclusively $\pi_0 = \pi_1 = \ldots = \pi_5 = 1/6$.