Problem 4.3.3

We consider a DTMC $\{X_n\}_{n \in \mathbb{N}_0}$ governed by

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}.$$

In order to use eq. (4.16), we need to calculate the transition probabilities $P_{00}^{(n)}$ for $n = 1, \dots, 5$. We can simply obtain these as the first entry of the matrix $P^{(n)} = P^n$.

n	1	2	3	4	5
$P_{00}^{(n)}$	0	1/4	1/8	3/8	7/32

We can now find the first return probabilities. By definition, $f_{00}^{(0)} = 0$ and $f_{00}^{(1)} = P_{00} = 0$. Using eq. (4.16), we get:

$$P_{00}^{(2)} = f_{00}^{(0)} P_{00}^{(2)} + f_{00}^{(1)} P_{00}^{(1)} + f_{00}^{(2)} P_{00}^{(0)}$$
$$= 0 \cdot 1/4 + 0 \cdot 0 + f_{00}^{(2)} \implies f_{00}^{(2)} = P_{00}^{(2)} = 1/4.$$

$$P_{00}^{(3)} = f_{00}^{(0)} P_{00}^{(3)} + f_{00}^{(1)} P_{00}^{(2)} + f_{00}^{(2)} P_{00}^{(1)} + f_{00}^{(3)} P_{00}^{(0)}$$
$$= 0 \cdot 1/8 + 0 \cdot 1/4 + 1/4 \cdot 0 + f_{00}^{(3)} \implies f_{00}^{(3)} = P_{00}^{(3)} = 1/8.$$

Similarly:

$$f_{00}^{(4)} = P_{00}^{(4)} - f_{00}^{(2)} P_{00}^{(2)} = 3/8 - (1/4)^2 = 6/16 - 1/16 = 5/16,$$

$$f_{00}^{(5)} = P_{00}^{(5)} - f_{00}^{(2)} P_{00}^{(3)} - f_{00}^{(3)} P_{00}^{(2)} = 7/32 - 1/4 \cdot 1/8 - 1/8 \cdot 1/4 = 5/32$$