Problem 4.1.13

We shall derive a classic result often used in the context of time-reversable Markov chains, namely that $\pi_i P_{ij}(r) = \pi_j P_{ji}$.

We consider the probability of interest and invokes Bayes' theorem.

$$\mathbb{P}(X_{n-1}=2 \mid X_n=1) = \mathbb{P}(X_n=1 \mid X_{n-1}=2)\mathbb{P}(X_{n-1}=2) / \mathbb{P}(X_n=1).$$
$$= P_{21}\, \mathbb{P}(X_{n-1}=2) / \mathbb{P}(X_n=1)$$

Now we take the limit as $n \longrightarrow \infty$ on both sides:

$$\lim_{n\to\infty} \mathbb{P}(X_{n-1}=2 \mid X_n=1) = P_{21}\pi_2/\pi_1.$$

In this case we find that $\pi_1 = 7/24$ and $\pi_2 = 6/24$.

Consequently, we get that

$$\lim_{n\to\infty} \mathbb{P}(X_{n-1}=2 \mid X_n=1) = 1/5 \cdot 6/24 \,/\, 7/24 = 6/35.$$