Problem 3.7.1

We are asked to find $(I-Q)^{-1} = W$ for a model with

$$\mathbb{P}(X_{n+1} = j \mid X_n = m) = \frac{1}{m}, \quad j = 0, 1, \ldots, m-1.$$

On p. ~~140~~ 141, we get the interpretation of W_{ij} as the expected number of visits to state j prior to absorption given initiation in state i.

Hence $W_{ij} = 0$ for $j > i$. Also, we have that $W_{ii} = 1$.

We can find the remaining entries of W using some simple recursions. (FSA)

$$W_{i,i-1} = \sum_{j=1}^{i} P_{ij} W_{j,i-1}.$$

We get:

$$W_{i,i-1} = 0 \cdot W_{i,i-1} + \frac{1}{i} \cdot 1 + \frac{1}{i} \cdot 0 + \ldots + \frac{1}{i} \cdot 0 = \frac{1}{i}.$$

$$W_{i,i-2} = 0 \cdot W_{i,i-2} + \frac{1}{i} \cdot W_{i-1,i-2} + \frac{1}{i} \cdot 1 + \frac{1}{i} \cdot 0 + \ldots + \frac{1}{i} \cdot 0$$

$$= \frac{1}{i}\left(W_{i-1,i-2} + 1\right) = \frac{1}{i}\left(W_{k,k-1} + 1\right)$$

$$= \frac{1}{i}\left(\frac{1}{i-1} + 1\right) = \frac{1}{i} \cdot \frac{i}{i-1} = \frac{1}{i-1}.$$

Problem 3.7.1

The results inspire the hypothesis
$W_{i,i-k} = (i-k+1)^{-1}$ for all i and $0 < k \leq i$.

We shall show this using a proof by induction. (with strong recurrence)

We showed the base case for $k=0$ and $k=1$.

Now, assume the hypothesis is true for all $0 < k \leq N$ for $0 < N \leq i$.

$$W_{i,i-(k+1)} = \sum_{j=i-(k+1)}^{i-1} P_{ij} W_{j,i-(k+1)}$$

$$= \frac{1}{i}\left(\sum_{j=i-(k+1)}^{i-1} W_{j,i-(k+1)}\right) = \frac{1}{i}\left(1 + \sum_{j=i-k}^{i-1} W_{j,i-(k+1)}\right)$$

$$= \frac{1}{i}\left(1 + \left[(i-1)-(i-k-1)\right](i-(k+1)+1)^{-1}\right)$$

$$= \frac{1}{i}\left(1 + \frac{k}{i-k}\right) = \frac{1}{i}\left(\frac{i}{i-k}\right) = \frac{1}{i-k},$$

which agrees with the hypothesis. In conclusion $W_{ij} = (j+1)^{-1}$.