Remember to include binomial representation.

Problem 3.9.4.

Let $\varphi(s) = 1 - p(1-s)^\beta$, where $0 < p, \beta < 1$.

We show that the iterates are

$$\varphi_n(s) = 1 - p^{1+\beta+\dots+\beta^{n-1}}(1-s)^{\beta^n}$$

by a proof by induction. We check the base case for $n=1$ and use weak induction.

$$\varphi(s) = 1 - p^1(1-s)^{\beta^1} = 1 - p(1-s)^\beta.$$ Checks out.

Now, assume that for an $n \in \mathbb{N}$, we have

$$\varphi_n(s) = 1 - p^{1+\beta+\dots+\beta^{n-1}}(1-s)^{\beta^n}.$$

Then for $n+1$, we have:

$$\varphi_{n+1}(s) = \varphi(\varphi_n(s)) = 1 - p(1 - (1 - p^{1+\beta+\dots+\beta^{n-1}}(1-s)^{\beta^n}))^\beta$$

$$= 1 - p(p^{1+\beta+\dots+\beta^{n-1}}(1-s)^{\beta^n})^\beta$$

$$= 1 - p(p^{1+\beta+\dots+\beta^{n-1}})^\beta((1-s)^{\beta^n})^\beta$$

$$= 1 - p \cdot p^{\beta+\beta^2+\dots+\beta^n}(1-s)^{\beta^{n+1}} = 1 - p^{1+\beta+\beta^2+\dots+\beta^n}(1-s)^{\beta^{n+1}},$$

which agrees with the hypothesis.

Problem 3.9.4.

We shall apply eq (3.104) to find the p.m.f. In order to do this, we need the derivatives of the p.g.f. We can establish these as

$$\varphi^{(n)}(s) = -p(-1)^n(1-s)^{\beta-n}\prod_{i=0}^{n-1}(\beta-i)$$

For $n=0$: $\varphi(s) = 1-p(1-s)^{\beta}$.

Hence $\varphi(0) = 1-p$ and $p_0 = \frac{1}{0!}\varphi(0) = 1-p$.

Obviously $0 < 1-p < 1$

For $n=1$: $\varphi^{(1)}(s) = p\cdot\beta\cdot(1-s)^{\beta-1}$.

Hence $\varphi^{(1)}(0) = p\beta$ and $p_1 = \frac{1}{1!}\varphi^{(1)}(0) = p\cdot\beta$.

Obviously $p_1 \in [0,1]$.

In general $p_k = \frac{1}{k!}\varphi^{(k)}(0) = \frac{1}{k!}\left(-p(-1)^k\prod_{i=0}^{k-1}(\beta-i)\right)$.

Moreover, $p_{k+1} = \frac{1}{(k+1)!}\left(-p(-1)^{k+1}\prod_{i=0}^{k}(\beta-i)\right)$

$$= \frac{1}{k+1}\cdot\frac{1}{k!}\left((-1)\cdot(-p)(-1)^k\prod_{i=0}^{k-1}(\beta-i)\cdot(\beta-k)\right)$$

$$= \frac{-1}{k+1}(\beta-k)\,p_k.$$

Problem 3.9.4

Note that for $k \geq 1$; we must have

that $k > \beta$. Hence

$$\frac{k-\beta}{k+1} \in [(1-\beta)/2; 1).$$

Thus, if , $0 \leq p_k \leq 1$, then $0 \leq p_k \frac{k-\beta}{k+1} \leq 1$,

meaning that $0 \leq p_{k+1} \leq 1$.

Finally, we evaluate the sum

$$\cancel{\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \left(-p\binom{\beta}{k}(-1)^k\right)}$$

$$\sum_{k=0}^{\infty} p_k = p_0 + \sum_{k=1}^{\infty} p_k = 1 - p + \sum_{k=1}^{\infty} \left(-p\binom{\beta}{k}(-1)^k\right)$$

$$= 1 - p - p\sum_{k=1}^{\infty} \binom{\beta}{k}(-1)^k = 1 - p - p\left(\sum_{k=0}^{\infty} \binom{\beta}{k}(-1)^k - 1\right)$$

$$= 1 - p + p - p\sum_{k=0}^{\infty} \binom{\beta}{k}(-1)^k = 1 - p(1 + (-1))^{\beta}$$

$$= 1.$$

Hence, it is a pmf.