Problem 8.4.6

NOTE: Some of you might have attempted to solve this problem by using theorem 8.3 with $A=0$ and $B=2$, however this formula is only valid for $0<A<1$ as a geometric Brownian motion can never attain the value zero. Even though this approach yields the correct result, this method is not valid.

Instead we solve the problem by recasting it in terms of a Brownian motion with drift. If we consider the geometric Brownian motion with drift parameter $a=0$, $\{Z_t\}_{t\geq 0}$, then

$$Z_t = Z_0 e^{X_t}, \quad (\text{eq. } 8.49)$$

where $\{X_t\}_{t\geq 0}$ is defined as $X_t = -\frac{1}{2}\sigma^2 t + \sigma B_t$ for a standard Brownian motion $\{B_t\}_{t\geq 0}$.
Consequently,

$$\mathbb{P}(\max Z_t > 2Z_0) = \mathbb{P}(\max Z_t/Z_0 > 2)$$

$$= \mathbb{P}(\max \ln(Z_t/Z_0) > \ln(2))$$

Here we have used that $Z_0 > 0$ and $Z_t > 0$, which implies that their ratio is positive.

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Furthermore, it is important that the logarithm is monotonically increasing. Now,

$$\ln(Z_t / Z_0) = \ln(Z_0 e^{X_t} / Z_0) = X_t,$$

which leads to

$$\mathbb{P}(\max Z_t > 2Z_0) = \mathbb{P}(\max X_t > \ln(2)).$$

Recall that $X_t = -\frac{1}{2}\sigma^2 t + \sigma B_t$, i.e. has drift $-\frac{1}{2}\sigma^2 < 0$ (as $\sigma^2 > 0$) and variance parameter σ.

Therefore, eq. (8.48) applies and yields

$$\begin{aligned}\mathbb{P}(\max X_t > \ln(2)) &= e^{-2|-\frac{1}{2}\sigma^2| \ln(2)/\sigma^2} \\ &= e^{-\ln(2)} \\ &= 2^{-1} = 1/2.\end{aligned}$$