Problem 8.3.3

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. As Brownian motion is a Gaussian process, cf. p. 398, it follows that (B_u, B_1) is jointly normally distributed. If F is their joint CDF such that

$$\mathbb{P}(B_u \leq x, B_1 \leq y) = F(x, y),$$

then we conclude that B_u and $B_1 \cdot u$ are jointly normally distributed as

$$\mathbb{P}(B_u \leq x, B_1 u \leq y) = \mathbb{P}(B_u \leq x, B_1 \leq y/u) = F(x, y/u)$$

for $\forall u > 0$. From sec. 8.1.4, we know that linear combinations of jointly normally distributed random variables are themselves jointly normally distributed. Consequently, $B_u - B_1 u$ and B_1 are jointly normally distributed. For jointly normally random variables, a covariance of zero implies independence as the joint PDF will factorize into the marginal laws. Hence, it is sufficient to show that $\mathrm{Cov}(B_u - uB_1, B_1) = 0$ to show independence between $B_u - uB_1$ and B_1.

Problem 8.3.3

Covariance is a bilinear operator and thus,

$$\begin{aligned} Cov(B_u - uB_1, B_1) &= Cov(B_u, B_1) - u\,Cov(B_1, B_1) \\ &= \min(u,1) - u \\ &= u - u = 0 \end{aligned}$$

as $0 < u < 1$. The result that $Cov(B_u, B_1)$ is $\min(u,1)$ is taken from eq. (8.10).
In conclusion, $B_u - uB_1$ and B_1 are independent

a) Without going into too much detail, the process $\{B_t - tB_1\}_{t\in[0,1]}$ is not a Brownian bridge per se, but it is equivalent to a Brownian bridge in law (in distribution).

To see this, note that $B_t = (B_t - tB_1) + tB_1$.
Consider then, for $0 < t < 1$,

$$\mathbb{P}(B_t \leq x \mid B_1 = 0) = \mathbb{P}((B_t - tB_1) + tB_1 \leq x \mid B_1 = 0).$$

Given $B_1 = 0$, $tB_1 = 0$, but $(B_t - tB_1)$ remains unaffected due to the independence. Hence,

$$\mathbb{P}(B_t \leq x \mid B_1 = 0) = \mathbb{P}(B_t - tB_1 \leq x).$$

In conclusion, for all $t \in (0,1)$, $\{B_t - tB_1\}$ is equal to a Brownian bridge in law.

Problem 8.3.3

b) Now that we have established that the process is indeed a Brownian bridge, we can use the construction to find the covariance function for the Brownian bridge. We shall again invoke the bilinearity of the covariance operator. We assume $0 < s < t < 1$:

$$\mathrm{Cov}(B_s - sB_1,\, B_t - tB_1)$$

$$= \mathrm{Cov}(B_s, B_t) - t\,\mathrm{Cov}(B_s, B_1) - s\,\mathrm{Cov}(B_1, B_t) + st\,\mathrm{Cov}(B_1, B_1)$$

$$= \mathrm{Cov}(B_s, B_t) - ts - st + st$$

$$= \mathrm{Cov}(B_s, B_t) - ts$$

$$= \min(s,t) - ts = s - ts = s(1-t).$$