Problem 8.4.2

This problem is very similar to the previous (8.4.1).

Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. We then define $X_t = B_t - at$, i.e. $\{X_t\}_{t\geq 0}$ is a Brownian motion with drift $-a$ and variance parameter $\sigma^2 = 1$. Since $a > 0$, the drift is negative.

Consider then the probability of interest.

$$\mathbb{P}\left(\max_{t\geq 0} (b + B_t)/(1+t) > a\right)$$

$$= \mathbb{P}\left(\max_{t\geq 0} (B_t - at - (a-b))/(1+t) > 0\right)$$

$$= \mathbb{P}\left(\max_{t\geq 0} (X_t - (a-b))/(1+t) > 0\right).$$

Note then that

$$\max_{t\geq 0}\left(\frac{X_t - (a-b)}{1+t}\right) > 0 \quad \text{iff.} \quad \max_{t\geq 0} (X_t - (a-b)) > 0.$$

Hence, as $a > b$, we can use eq. (8.48):

$$\mathbb{P}\left(\max_{t\geq 0} (b+B_t)/(1+t) > a\right) = \mathbb{P}\left(\max_{t\geq 0} X_t - (a-b) > 0\right)$$

$$= \mathbb{P}\left(\max_{t\geq 0} X_t > a-b\right) = e^{-2|-a|(a-b)} = e^{-2a(a-b)}.$$