We define $\{B_t\}_{t\geq 0}$ as a standard Brownian motion and $\{M_t\}_{t\geq 0}$ as the maximum process of $\{B_t\}$. We want to evaluate $\mathbb{P}(M_t \geq z, B_t \leq x)$. Consider then all sample paths that satisfy the conditions $M_t \geq z$ and $B_t \leq x$ for a fixed $t > 0$.

By the continuity of the sample paths (almost surely continuous paths), we know that there exist a time $\tau$ such that $B_\tau = z$. All sample paths satisfying the two conditions, will then decrease with at least $z - x$ (recall that $z > x$) from $\tau$ to $t$. ($\tau < t$). For each of these paths, we consider the reflected path $B_t^*$ as defined in sec. 8.2.1 on top of p. 406, where we from time $\tau$ reflect the path about the line $y = z$. The reflected path reach $B_\tau = z$ and the increases at least $z - x$ from $\tau$ to $t$. This implies that $B_t^* \geq z + (z - x)$. Using the reflection principle, we know that the original sample path is equally likely as the reflected sample path, i.e.

minus

$$\mathbb{P}(M_t \geq z, B_t \leq z - (z + x))$$

$$= \mathbb{P}(M_t \geq z, B_t \geq z + (z - x)) = \mathbb{P}(B_t \geq 2z - x).$$

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The last equality is due to the fact that if $B_t \geq 2z - x$, then $M_t \geq z$, as $0 < x < z$.

Now, we know that

$$\mathbb{P}(B_t \geq 2z - x) = 1 - \mathbb{P}(B_t < 2z - x)$$
$$= 1 - \Phi((2z - x)/t^{\frac{1}{2}}),$$

where $\Phi$ is the CDF of a standard normal distribution. The reason we divide by $t^{\frac{1}{2}} = \sqrt{t}$ is because we standardize $B_t \sim \mathcal{N}(0, t)$ to $B_t/\sqrt{t} \sim \mathcal{N}(0,1)$.

In summary,

$$\mathbb{P}(M_t \geq z, B_t \leq x) = 1 - \Phi((2z - x)/\sqrt{t}).$$

Now, let $f_{M_t}(z)$ and $f_{B_t}(x)$ be the density functions of $M_t$ and $B_t$, respectively. We also define $f_t(z, x)$ as the joint density function of $(M_t, B_t)$. Then, we have

$$\mathbb{P}(M_t \geq z, B_t \leq x) = \int_z^\infty \int_{-\infty}^x f_t(u, w)\,dw\,du.$$

Thus,

$$\frac{\partial^2}{\partial z \partial x} \mathbb{P}(M_t \geq z, B_t \leq x) = \frac{\partial^2}{\partial z \partial x} \int_z^\infty \int_{-\infty}^x f_t(u, w)\,dw\,du$$

$$= \frac{\partial}{\partial z} \int_z^\infty \left( \frac{\partial}{\partial x} \int_{-\infty}^x f_t(u, w)\,dw \right) du$$

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The fundamental theorem of calculus shows that

$$\frac{\partial}{\partial x}\int_{-\infty}^{x} f_t(u,w)\,dw = f_t(u,x).$$

Hence,

$$\frac{\partial}{\partial z}\int_{z}^{\infty}\left(\frac{\partial}{\partial x}\int_{-\infty}^{x} f_t(u,w)\,dw\right)du$$

$$= \frac{\partial}{\partial z}\int_{z}^{\infty} f_t(u,x)\,du$$

$$= -f_t(z,x).$$

In conclusion, we can find the joint pdf as $f_t(z,x) = -\partial^2/\partial z\partial x\, \mathbb{P}(M_t \geq z, B_t \leq x)$.

We take the partial derivative of $\mathbb{P}(M_t \geq z, B_t \leq x)$ w.r.t. $x$:

$$\frac{\partial}{\partial x}\mathbb{P}(M_t \geq z, B_t \leq x) = \frac{\partial}{\partial x}\left(1 - \Phi((2z-x)/\sqrt{t}\right)$$

$$= -\varphi((2z-x)/\sqrt{t})(-1/\sqrt{t}) = \varphi((2z-x)/\sqrt{t})\cdot 1/\sqrt{t}.$$

We then take the partial derivative with respect to $z$:

$$\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\mathbb{P}(M_t \geq z, B_t \leq x)\right) = \frac{\partial^2}{\partial z\partial x}\mathbb{P}(M_t \geq z, B_t \leq x)$$

$$= \frac{\partial}{\partial z}\left(\varphi\left(\frac{2z-x}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}\right) = \varphi'\left(\frac{2z-x}{\sqrt{t}}\right)\frac{2}{\sqrt{t}}\frac{1}{\sqrt{t}}.$$

In both cases, we have applied the chain rule.

Here $\varphi$ is the PDF of a standard normal distribution. We can find the derivative of the function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\varphi'(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx}\left(e^{-\frac{1}{2}x^2}\right) = \frac{1}{\sqrt{2\pi}}\left(-x e^{-\frac{1}{2}x^2}\right)$$

$$= -x\varphi(x).$$

Finally, we get that

$$-\frac{\partial^2}{\partial z \partial x} \mathbb{P}(M_t \geq z, B_t \leq x)$$

$$= -\varphi'\left(\frac{2z-x}{\sqrt{t}}\right)\frac{2}{t}$$

$$= -\left(-\frac{2z-x}{\sqrt{t}}\varphi\left(\frac{2z-x}{\sqrt{t}}\right)\right)\frac{2}{t}$$

$$= \varphi\left(\frac{2z-x}{\sqrt{t}}\right)\left(\frac{2z-x}{\sqrt{t}}\right)\frac{2}{t}, \quad 0 < x < z$$