Exercise 8.1.6

Let $\{B_t : t \geq 0\}$ be a standard Brownian motion.

a) First note that $B(u+v) = B_u + \Delta B_v$, where the two RVs are independent.

$B_u \sim \mathcal{N}(0, u)$ and $\Delta B_v \sim \mathcal{N}(0, v)$.

Thus: $B(u) + B(u+v) = 2B_u + \Delta B_v$.

Since sums of normally distributed (independent) RVs are normally distributed, we have that

$2B_u + \Delta B_v \sim \mathcal{N}(\mu, \sigma^2)$.

Here:

$\mu = \mathbb{E}[2B_u + \Delta B_v] = \mathbb{E}[2B_u] + \mathbb{E}[\Delta B_v] = 2 \cdot 0 + 0 = 0.$

$\sigma^2 = \mathbb{V}[2B_u + \Delta B_v] = \mathbb{V}[2B_u] + \mathbb{V}[\Delta B_v] + 2\,\mathrm{Cov}(2B_u, \Delta B_v)$

(independence) $= 2^2 \mathbb{V}[B_u] + \mathbb{V}[\Delta B_v]$

$= 4u + v.$

Hence $B(u) + B(u+v) \sim \mathcal{N}(0, 4u + v)$.

Exercise 8.1.6 (2)

b) By similar arguments:

$$B(u) + B(u+v) + B(u+v+w)$$

$$= B(u) + (B(u) + \Delta B_v) + (B(u+v) + \Delta B_w)$$

$$= B_u + (B_u + \Delta B_v) + (B_u + \Delta B_v + \Delta B_w)$$

$$= 3B_u + 2\Delta B_v + \Delta B_w \sim \mathcal{N}(\mu, \sigma^2).$$

$$\mu = \mathbb{E}[3B_u + 2\Delta B_v + \Delta B_w] = 3\mathbb{E}[B_u] + 2\mathbb{E}[\Delta B_v] + \mathbb{E}[\Delta B_w]$$

$$= 0,$$

$$\sigma^2 = \mathbb{V}[3B_u + 2\Delta B_v + \Delta B_w] = 3^2\mathbb{V}[B_u] + 2^2\mathbb{V}[\Delta B_v] + \mathbb{V}[\Delta B_w]$$

due to independence. Thus,

$$\sigma^2 = 9u + 4v + w$$

Hence:

$$B(u) + B(u+v) + B(u+v+w) \sim \mathcal{N}(0, 9u + 4v + w).$$