Exercise 8.1.4.

Let $\{B_t : t \geq 0\}$ be a standard Brownian motion and denote $B(s+t) - B(s) = \Delta B_t$ for all s.

a) Note that

$$B(u+v) = B(u) + B(u+v) - B(u) = B_u + \Delta B_v,$$
$$B(u+v+w) = B(u+v) + \Delta B_w = B_u + \Delta B_v + \Delta B_w$$

Hence;

$$\mathbb{E}[B(u)B(u+v)B(u+v+w)]$$

$$= \mathbb{E}[B_u(B_u + \Delta B_v)(B_u + \Delta B_v + \Delta B_w)]$$

$$= \mathbb{E}[(B_u^2 + B_u \Delta B_v)(B_u + \Delta B_v + \Delta B_w)]$$

$$= \mathbb{E}[B_u^3 + B_u^2(\Delta B_v + \Delta B_w) + B_u^2 \Delta B_v + B_u \Delta B_v^2 + B_u \Delta B_v \Delta B_w$$

Then invoke the following:

1) Linearity of the expectation operator

2) $\mathbb{E}[B_u^3] = \mathbb{E}[B_u] = 0$ and $\mathbb{E}[\Delta B_v] = \mathbb{E}[\Delta B_w] = 0$

3) $\mathbb{E}[B_u^2] = \mathbb{V}[B_u] = u$ and $\mathbb{E}[\Delta B_t^2] = \mathbb{V}[\Delta B_t] = t$

Exercise 8.1.4 (2)

$$\mathbb{E}[B(u)B(u+v)B(u+v+w)] = 0$$

b) Using the same arguments as in a)

$$\mathbb{E}[B(u)B(u+v)B(u+v+w)B(u+v+w+x)]$$

$$= \mathbb{E}[B_u^4] + 3\mathbb{E}[B_u^2]\mathbb{E}[\Delta B_v^2] + \mathbb{E}[B_u^2]\mathbb{E}[\Delta B_w^2]$$

To calculate $\mathbb{E}[B_u^4]$ note that $B_u/\sqrt{u} \sim \mathcal{N}(0,1)$.

Hence: $u^2\mathbb{E}[(B_u/\sqrt{u})^4] = \mathbb{E}[B_u^4]$ implies that

$$\mathbb{E}[B_u^4] = u^2 \frac{4!}{4 \cdot 2!} = 3u^2.$$

In conclusion:

$$\mathbb{E}[B(u)B(u+v)B(u+v+w)B(u+v+w+x)]$$

$$= 3u^2 + 3uv + uw.$$