Problem 8.1.1

We consider a simple random walk $\{S_n\}_{n\in\mathbb{N}_0}$ with $S_0 = 0$. The process is defined as $S_n = \sum_{i=1}^{n} \varepsilon_i$, $n \geq 1$, where the summands are independent and identically distributed with $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = 1/2$.

Note that $\mathbb{E}[\varepsilon_i] = 1/2 \cdot 1 + 1/2 \cdot (-1) = 0$ and $\mathbb{V}[\varepsilon_i] = \mathbb{E}[\varepsilon_i^2] - \mathbb{E}[\varepsilon_i]^2 = \mathbb{E}[\varepsilon_i^2] = 1/2 \cdot (1)^2 + 1/2 \cdot (-1)^2 = 1$.

We now define $T = \min\{t \geq 0 : B_t = -a \vee B_t = b\}$, where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion, i.e. $B_0 = 0$, $\sigma = 1$.

In order to evaluate $\mathbb{E}[T]$, we want to define a sequence $\{T_n\}_{n\in\mathbb{N}_0}$ which converges in distribution to T (as T is a random variable). If $T_n \xrightarrow{d} T$ (in distribution) then $\mathbb{E}[T_n] \to \mathbb{E}[T]$ weakly (in the Hilbert space sense).

We let $T_n = \min\{t \geq 0 : B_t^n \in \{-a, b\}\}$, where B_t^n is formed as in eq. (8.11), i.e. $B_t^n = S_{\lfloor nt \rfloor}/\sqrt{n}$, for $t \geq 0$. We can then rewrite the expression for T_n as follows

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$$T_n = \min\{t \geq 0 : B^n_t \in \{-a, b\}\}$$

$$= \min\{t \geq 0 : S_{[nt]}/\sqrt{n} \in \{-a, b\}\} \quad (\text{by } (8.11))$$

$$= \min\{t \geq 0 : S_{[nt]} \in \{-a\sqrt{n}, b\sqrt{n}\}\}.$$

Note that $[nt] = k$ implies that $k/n \leq t < k/n + 1/n$. Therefore, the minimal t satisfying $[nt] = k$ is $t = k/n$. Thus,

$$T_n = \frac{1}{n}\min\{k \geq 0 : S_k \in \{-a\sqrt{n}, b\sqrt{n}\}\}.$$

Now, we can invoke eq. (3.52), which is the result given in the problem description;

$$\mathbb{E}[\min\{k \geq 0 : S_k \in \{-a\sqrt{n}, b\sqrt{n}\}\}] = abn.$$

Hence $\mathbb{E}[T_n] = \frac{1}{n} \cdot abn = ab.$

In conclusion, per the invariance principle $B^n_t \xrightarrow{d} B_t$ (weakly or in distribution), and therefore $T_n \xrightarrow{d} T$. We therefore know that $\mathbb{E}[T_n] \xrightarrow{w} \mathbb{E}[T]$. Hence

$$\mathbb{E}[T] = \lim_{n \to \infty} \mathbb{E}[T_n] = \lim_{n \to \infty} ab = ab.$$