Exercise 8.1.5

Without loss of generality, we can assume that $0 \le s < t$.

a) $U(t) = e^{-t} B(e^{2t})$, $t \ge 0$.

$\mathrm{Cov}(U(s), U(t)) = \mathbb{E}[U(s)U(t)] - \mathbb{E}[U(t)]\mathbb{E}[U(s)]$.

We find that

$\mathbb{E}[U(t)] = \mathbb{E}[e^{-t} B(e^{2t})] = e^{-t}\mathbb{E}[B(e^{2t})] = 0$

since $B(e^{2t}) \sim \mathcal{N}(0, e^{2t})$. Thus; $\mathbb{E}[U(s)] = 0$.

$$\mathbb{E}[U(s)U(t)] = \mathbb{E}[e^{-s} B(e^{2s}) e^{-t} B(e^{2t})]$$

$$= e^{-(s+t)} \mathbb{E}[B(e^{2s}) B(e^{2t})]$$

$$= e^{-(s+t)} e^{2s} = e^{s-t}.$$

More generally $e^{-|s-t|}$.

b) $V(t) = (1-t) B(t/(1-t))$.

We assume that $0 \le s < t < 1$.

Exercise 8.1.5 (2)

Note that for $0 \leq t < 1$

$$\mathbb{E}[V(t)] = \mathbb{E}[(1-t)B(t/(1-t))]$$

$$= (1-t)\mathbb{E}[B(t/(1-t))] = 0$$

Since $B(t/(1-t)) \sim \mathcal{N}(0, t/(1-t))$.

Furthermore, we need the following

$$s < t \implies 1/t < 1/s \implies 1/t - 1 < 1/s - 1$$

$$\implies 1/(1/s - 1) < 1/(1/t - 1)$$

$$\implies s/(1-s) < t/(1-t).$$

Then:

$$\mathrm{Cov}(V(s), V(t)) = \mathbb{E}[V(s)V(t)] - \mathbb{E}[V(s)]\mathbb{E}[V(t)]$$

$$= \mathbb{E}[V(s)V(t)]$$

$$= \mathbb{E}[(1-s)B(s/(1-s))(1-t)B(t/(1-t))]$$

$$= (1-s)(1-t)\mathbb{E}[B(s/(1-s))B(t/(1-t))]$$

$$= (1-s)(1-t)s/(1-s) = (1-t)s.$$

Exercise 8.1.5 (3)

c) $W(t) = tB(1/t),\ t > 0.$

Again we assume $0 < s < t$.

$\mathrm{Cov}(W(s), W(t)) = \mathbb{E}[W(s)W(t)] - \mathbb{E}[W(s)]\mathbb{E}[W(t)]$

(By similar arguments) $= \mathbb{E}[W(s)W(t)]$

$= st\,\mathbb{E}[B(1/s)B(1/t)]$

$= st \cdot 1/t = s,$

or more generally $\min\{s, t\}$.