Problem 8.1.2

Recall that $B_t \sim N(0,t)$, as $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. Let f_B denote the density function of B_t, i.e

$$f_B(x) = (2\pi t)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2/t) \text{ for } x \in \mathbb{R}$$

Hence

$$\mathbb{E}[\exp(\lambda B_t)] = \int_{-\infty}^{\infty} \exp(\lambda x) f_B(x)\,dx$$

$$= \int_{-\infty}^{\infty} \exp(\lambda x)(2\pi t)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2/t)\,dx$$

$$= \int_{-\infty}^{\infty} (2\pi t)^{-\frac{1}{2}} \exp(\lambda x - \tfrac{1}{2}x^2/t)\,dx.$$

We then rewrite the exponent. We sort of "complete the square".

$$-\tfrac{1}{2}x^2/t + \lambda x = -\tfrac{1}{2}(x^2/t - 2\lambda x) = -\tfrac{1}{2t}(x^2 - 2\lambda t x)$$
$$= -\tfrac{1}{2t}\left((x-\lambda t)^2 - (\lambda t)^2\right)$$

Inserting this in the above integral yields

$$\mathbb{E}[\exp(\lambda B_t)] = \int_{-\infty}^{\infty} (2\pi t)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}(x-\lambda t)^2/t) \cdot \exp(+\tfrac{1}{2t}(\lambda t)^2)\,dx$$

$$= \exp(+\tfrac{1}{2t}(\lambda t)^2) \int_{-\infty}^{\infty} (2\pi t)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}(x-\lambda t)^2/t)\,dx$$

Problem 8.1.2

Notice that the integrand is the density function of random variable following a $\mathcal{N}(\lambda t, t)$-distribution. Integrating a density function over its entire domain yields one. Hence,

$$\mathbb{E}[\exp(\lambda B_t)] = \exp\left(+\frac{1}{2t}(\lambda t)^2\right)$$

$$= \exp(\lambda^2/2t).$$