Problem 8.1.7

The definition of a Martingale includes 3 conditions of which the third concerns adaption and measurability. This is beyond the scope of our course, so we shall only look at the two first conditions as given in sec. 2.5.1. The conditions for $\{X_n\}_{n\in\mathbb{N}_0}$ being a Martingale are:

1) $\mathbb{E}[|X_n|]$ is finite for all $n\in\mathbb{N}_0$,

2) $\mathbb{E}[X_{n+1}\,|\,X_0,\dots,X_n] = X_n$ for all $n\in\mathbb{N}_0$.

a) We check the two conditions for the process $\{B_n\}_{n\in\mathbb{N}_0}$, where $B_n \sim \mathcal{N}(0,n)$. Consequently, $|B_n|$ follows a folded normal distribution, which has finite mean. Specifically, let $f_B(x)$ be the PDF of B_n, then

$$\begin{aligned}\mathbb{E}[|B_n|] &= \int_{-\infty}^{\infty} |x|\, f_B(x)\,dx \\ &= 2\int_0^{\infty} x\, f_B(x)\,dx \qquad (\text{symmetry of a } \mathcal{N}\text{-PDF}) \\ &= \sqrt{2n/\pi}.\end{aligned}$$

The second condition is also satisfied as $(B_{n+1}-B_n)+B_n = B_{n+1}$ and $B_{n+1}-B_n$ is independent of the past, cf. p. 394 below the definition.

Problem 8.1.7

Hence

$\mathbb{E}[B_{n+1} \mid B_0, \ldots, B_n]$

$= \mathbb{E}[(B_{n+1} - B_n) + B_n \mid B_0, \ldots, B_n]$

$= \mathbb{E}[B_{n+1} - B_n \mid B_0, \ldots, B_n] + \mathbb{E}[B_n \mid B_0, \ldots, B_n]$

$= 0 + B_n = B_n.$

as increments of standard Brownian motions have expectation zero. In conclusion, $\{B_n\}_{n \in \mathbb{N}_0}$ is a Martingale.

b) We now analyse the process $\{B_n^2 - n\}_{n \in \mathbb{N}_0}$

First note that $\mathbb{V}[B_n] = \mathbb{E}[B_n^2] - \mathbb{E}[B_n]^2 = \mathbb{E}[B_n^2]$. We know that $\mathbb{V}[B_n] = n$ and so $\mathbb{E}[B_n^2] = n$.

As a consequence, $\mathbb{E}[B_n^2 - n] = 0$.

BUT, we are asked to find $\mathbb{E}[|B_n^2 - n|]$ and we cannot just assume that $\mathbb{E}[|B_n^2 - n|] = |\mathbb{E}[B_n^2 - n]|$.

Instead, since both $\mathbb{E}[|B_n^2|]$ and $\mathbb{E}[|-n|]$ are finite, Minkowski's inequality for $p=1$ applies and assures that

$$\mathbb{E}[|B_n^2 - n|] \leq \mathbb{E}[|B_n^2|] + \mathbb{E}[|-n|]$$
$$= \mathbb{E}[B_n^2] + \mathbb{E}[n]$$
$$= n + n = 2n.$$

As $\mathbb{E}[|B_n^2 - n|] \leq 2n$, we know it is finite and $\{B_n^2 - n\}$ satisfies the first condition. For the second condition, we get the wanted result by straigthforward calculations:

$$\mathbb{E}[B_{n+1}^2 - (n+1) | B_n^2 - n]$$
$$= \mathbb{E}[B_{n+1}^2 | B_n^2 - n] + (n+1) \quad ((n+1) \text{ is a constant})$$
$$= \mathbb{E}[(B_{n+1} - B_n + B_n)^2 | B_n^2 - n] - (n+1)$$
$$= \mathbb{E}[(B_{n+1} - B_n)^2 + 2(B_{n+1} - B_n)B_n + B_n^2 | B_n^2 - n] - (n+1).$$

Conditional expectation is a linear operator and hence;

$$\mathbb{E}[B_{n+1}^2 - (n+1) | B_n^2 - n]$$
$$= \mathbb{E}[(B_{n+1} - B_n)^2 | B_n^2 - n] + 2\mathbb{E}[(B_{n+1} - B_n)B_n | B_n^2 - n]$$
$$+ \mathbb{E}[B_n^2 | B_n^2 - n] - (n+1).$$

Since $B_{n+1} - B_n$ is independent of the past and notably B_n (and thus also $B_n^2 - n$), we can simplify the terms

$$\mathbb{E}[(B_{n+1} - B_n)^2 \mid B_n^2 - n] = \mathbb{E}[(B_{n+1} - B_n)^2],$$

$$\mathbb{E}[(B_{n+1} - B_n) B_n \mid B_n^2 - n] = \mathbb{E}[(B_{n+1} - B_n)] B_n,$$

we can treat B_n as known since we condition on $B_n^2 - n$, and finally,

$$\mathbb{E}[B_n^2 \mid B_n^2 - n] = B_n^2.$$

Now, recall that $B_{n+1} - B_n \sim \mathcal{N}(0, (n+1) - n)$, i.e. a standard normal distribution, and therefore we have that $\mathbb{E}[B_{n+1} - B_n] = 0$ and $\mathbb{E}[(B_{n+1} - B_n)^2] = \mathbb{V}[B_{n+1} - B_n] + \mathbb{E}[B_{n+1} - B_n]^2$, which is 1. In conclusion,

$$\mathbb{E}[B_{n+1}^2 - (n+1) \mid B_n^2 - n]$$
$$= 1 + 2 \cdot 0 \cdot B_n + B_n^2 - (n+1) = B_n^2 - n.$$

Thus, $\{B_n^2 - n\}$ satisfies the second condition, and we conclude that the process is indeed a Martingale.